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Optimization of the multigrid-convergence rate on semi-structured meshes by local Fourier analysis

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ABSTRACT

In this paper a local Fourier analysis for multigrid methods on tetrahedral grids is presented. Different smoothers for the discretization of the Laplace operator by linear finite elements on such grids are analyzed. A four-color smoother is presented as an efficient choice for regular tetrahedral grids, whereas line and plane relaxations are needed for poorly shaped tetrahedra. A novel partitioning of the Fourier space is proposed to analyze the four-color smoother. Numerical test calculations validate the theoretical predictions. A multigrid method is constructed in a block-wise form, by using different smoothers and different numbers of pre- and post-smoothing steps in each tetrahedron of the coarsest grid of the domain. Some numerical experiments are presented to illustrate the efficiency of this multigrid algorithm.

1. Introduction and terminology

Multigrid (MG) methods [1–4] are known to have optimal computational complexity for solving many numerical problems. However, in practice their performance in solving individual problems varies significantly. The speed of convergence towards a solution depends on the numerical properties of the underlying problem, e. g., the type of a differential equation and the method used for discretizing the equation. Besides that, the user can choose from a variety of algorithms for the components of the MG method, most prominently the smoother, the restriction, and the prolongation. Choosing the appropriate components for a specific problem has also a great impact on the overall performance.

For solving elliptic PDEs, finite element methods are often preferred over other discretization schemes, because they permit the use of flexible, unstructured meshes. Algebraic multigrid (AMG) methods [5–7,3] inherently support unstructured meshes by construction. BoomerAMG from the Hypre package is a popular implementation of AMG [8]. Geometric multigrid, in contrast, relies on a given hierarchy of nested meshes. Geometric multigrid may achieve a significantly higher performance than the algebraic multigrid in terms of unknowns computed per second.

The Hierarchical Hybrid Grids (HHGs) software framework [9,10] is designed to close this gap between finite element flexibility and geometric multigrid performance by using a compromise between structured and unstructured grids. A coarse input finite element mesh is organized into grid primitive vertices, edges, faces, and volumes. The primitives are then refined in a structured way, as indicated in Fig. 1 for the two-dimensional case. In the case of tetrahedral grids, Bey's refinement strategy will be considered [11]. There each input tetrahedron is subdivided into eight child tetrahedra of equal volume, in such a way that each corner of a child coincides to either a corner or an edge midpoint of the parent. The accuracy of tetrahedral finite elements w.r.t. their maximal angle is discussed e.g. in [12]. The HHG data layout preserves the flexibility of

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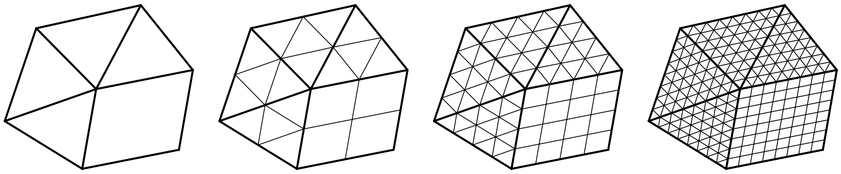

Fig. 1. Regular refinement example for a two-dimensional input mesh. Beginning with the input mesh on the left, each successive level of refinement creates a new mesh that has a larger number of interior points with structured couplings.

unstructured meshes, while the regular internal structure of the primitives allows for an efficient implementation on current computer architectures, especially on parallel computers. In parallel runs on up to 294 912 cores, HHG has demonstrated excellent performance for solving linear systems with up to 10^{12} unknowns [13]. Semi-structured meshes also support a local tuning of the smoothing parameters as will be discussed later in this paper.

Generally the design of MG algorithms and their components should be based on a careful performance analysis. Such an analysis could be based in the comparison of convergence rates of different algorithmic variants. For standard cases, such as the 7-point stencil for the Laplace operator in 3D, convergence rates are reported in the literature. In our case, however, we will use 15-point stencils as they arise in the HHG framework with tetrahedral finite elements.

Local Fourier analysis (LFA) [14] is a very useful tool to predict the asymptotic convergence factors of MG methods with high accuracy quantitatively. Therefore it is widely used to design efficient MG algorithms. In the LFA an infinite regular grid is considered and boundary conditions are ignored. On an infinite grid, the discrete solution and the corresponding error can be represented by linear combinations of certain complex exponential functions, the Fourier modes, which form a unitary basis of the space of grid functions with bounded l_2-norm. The LFA monograph of Wienands and Joppich [15] provides an excellent background for experimenting with Fourier analysis. Recent advances in this context include LFA for hexagonal meshes [16], multigrid as a preconditioner [17], optimal control problems [18], and discontinuous Galerkin discretizations [19]. In [20], an LFA for multigrid methods for the finite element discretization of the two-dimensional curl–curl equation on a quadrilateral grid has been introduced. Recently a generalization of the LFA to triangular grids has been proposed in [21] for a two-dimensional scalar problem. The key to carrying out this generalization is to express the Fourier transform in a special coordinate system. Our aim in this paper is to show that it is possible to perform an LFA also on three-dimensional tetrahedral grids so that we can use it to design efficient multigrid solvers in the HHG framework. To choose suitable components of the multigrid method, the LFA will be applied for each tetrahedron of the input mesh in such a way that the global behavior of the method becomes as efficient as possible.

In a multigrid algorithm, the smoother has the task to reduce the high-frequency error components, while the coarse-grid correction reduces the low-frequency components. Both, smoother and coarse-grid correction [22,23] can be tuned to optimally suit the numerical problem, i. e. the differential equation and its discretization. In this paper, we focus on optimizing the smoother depending on the shape of the tetrahedra. Besides choosing the smoother type, two more parameters can be changed to achieve more efficient smoothing behavior: the under-/over-relaxation parameter (ω) and the number of pre-/post-smoothing steps per multigrid cycle (ν_1, ν_2). All the parameters can even be adapted locally to the problem, if the numerical properties differ strongly across the domain.

The Gauss–Seidel algorithm often is a better smoother than the Jacobi algorithm, but due to its data dependencies, it is potentially slower. The data dependencies can be eliminated by coloring the grid points such that points of the same color are not directly connected to each other, and can thus be updated in parallel. In a 2D rectangular grid with a 5-point discretization stencil, e.g. two colors are needed, whereas the 15-point stencil within HHG solver requires four colors, see Fig. 2. Note that a relaxation parameter ω can be chosen individually for each of the colors ($\omega_i = \omega_1, \omega_2, \omega_3, \omega_4$).

In this work, a linear finite element discretization of the Laplace equation will be analyzed and, as we will see, four-color relaxation results in a very efficient smoother for regular tetrahedral grids. However, when poorly shaped tetrahedra occur, then point-wise smoothers are not efficient anymore. In this case we will consider block-wise smoothers. In particular, line- and plane-wise smoothers will be used. Note that for Bey refinement, seven different line smoothers can be defined that correspond with the seven directions that appear in the connections between each pair of stencil entries, see Fig. 2. Similarly, seven different plane smoothers can be considered. Four faces have the orientation of the un-refined tetrahedron face. Three other face orientations can be spanned up by two vectors connecting two opposing edges. The remaining multigrid components will be standard, i.e. we will use linear interpolation and its adjoint as transfer operators and will use Galerkin coarsening for defining the coarse-grid operators.

The organization of the remaining paper is as follows. In Section 2 we present the local Fourier analysis for tetrahedral grids. The key here is to consider a basis of the function space appropriate for the structure of the grid. In Section 3, a four-color smoother is defined for which smoothing and two-grid analysis will be performed. In this study, an important aspect is to show the decomposition of the Fourier space into minimal invariant subspaces. These are four-dimensional for the smoothing analysis and have sixteen dimensions for the two-grid analysis.

Section 4 is devoted to Fourier analysis results for the Laplace operator discretized by linear finite elements on tetrahedral grids. Different smoothers will be proposed depending on the particular geometry of the grid on each tetrahedral patch. It will be shown that a four-color smoother will be the best choice for regular tetrahedra. Otherwise block smoothers may be