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Improving problem reduction for 0–1 Multidimensional Knapsack Problems with valid inequalities

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ABSTRACT

This paper investigates the problem reduction heuristic for the Multidimensional Knapsack Problem (MKP). The MKP formulation is first strengthened by the Global Lifted Cover Inequalities (GLCI) using the cutting plane approach. The dynamic core problem heuristic is then applied to find good solutions. The GLCI is described in the general lifting framework and several variants are introduced. A Two-level Core problem Heuristic is also proposed to tackle large instances. Computational experiments were carried out on classic benchmark problems to demonstrate the effectiveness of this new method.

1. Introduction

The Multidimensional Knapsack Problem (MKP) is an extension of the classic Knapsack Problem (KP) with more than one knapsack constraints. Given m knapsacks with capacities b_i, $i=1,\dots,m$, and n items which require resource consumption of $a_{i,j}$ units in the i-th knapsack ($i=1,\dots,m$), and yield c_j units of profit upon inclusion for item j, $j=1,\dots,n$, the goal is to find a subset of items that yields maximum profit, denoted by z^*, without exceeding the knapsack capacities. The MKP can be defined by the following Integer Linear Programming (ILP):

$$(\text{MKP}) \quad z^* = \max\{c^T x; Ax \le b, x \in \{0,1\}^n\} \tag{1}$$

where $c=[c_1,c_2,\dots,c_n]^T$ is an n-dimensional vector of profits, $x=[x_1,x_2,\dots,x_n]^T$ is an n-dimensional vector of 0–1 decision variables indicating whether an item is included or not, $A=[a_{i,j}]$, $i=1,2,\dots,m, j=1,2,\dots,n$, is an $m \times n$ coefficient matrix of resource requirements, and $b=[b_1,b_2,\dots,b_m]^T$ is an m-dimensional vector of resource capacities. It is further assumed that all parameters are non-negative integers.

The MKP is a well-studied, strongly NP-hard combinatorial optimisation problem, and has found applications in many practical areas involving resource allocation. An early review of the MKP was given by [1], and a comprehensive overview of practical and theoretical results can be found in the monograph on knapsack problems [2]. Excellent reviews on solution methods and practical applications can be found in [3,4]. In spite of the tremendous progress made by commercial ILP solvers, the methods currently yielding the best results, at least for commonly used benchmark instances in [5], are mainly from the specialised algorithms [6–10]. The main drawback of these approaches is, however, the huge running time for the large instances in the OR-Library [11].

Among the fast heuristics aiming for satisfactory solutions, the core problem based approach has been shown to be very competitive for its simplicity and efficiency. The core concept was first presented for the classical knapsack problem in [12], and extended later for MKP in [13]. The main idea is to reduce the original problem to a core of items for which it is hard to decide whether or not they will occur in an optimal solution, whereas all variables corresponding to items outside the core are fixed to their presumably optimal values. The core problem based heuristics typically determine an approximate core by calculating some simple *efficiency measures* for each variable. Various efficiency measures were proposed and compared in [14] in terms of core size and accuracy. It concluded that the core problem heuristic, especially with the efficiency measures exploiting the dual values of the Linear Programming (LP) relaxation of MKP, can yield highly competitive results in significantly shorter run-times. Recently a new efficiency measure based on the reduced cost of the LP relaxation of MKP was proposed in [15]. Instead of fixed core sizes commonly used in previous literature, this novel approach can adaptively change the core size for each instance. Comprehensive experimentation demonstrates that this approach performs consistently well on well-designed sets of test cases.

Since the LP relaxation plays an important role in the most successful core problem based heuristics for MKP, this inspires us to investigate in this paper the effectiveness of strengthening the LP relaxation of MKP with valid inequalities, with the hope of better performance on the hard instances. Although cuts are

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commonly used in the branch and cut algorithms for general ILP solvers, our application allows for classes of cuts with more expensive computational costs.

Lifted Cover Inequalities (LCI), among other valid inequalities for 0–1 knapsack polytopes, have proven useful when tackling hard 0–1 Integer Programming problems including the MKP [16,17]. Recently the Global LCI (GLCI) [18] was proposed for MKP to take into consideration multiple knapsack constraints simultaneously by solving LPs to lift the coefficients of a valid inequality. Although the GLCI may not even define a face of MKP, it can still be stronger than LCI, especially for MKP with many knapsack constraints. Nevertheless the GLCI was not evaluated for the effectiveness in the branch and bound algorithm. In this paper we apply the GLCI and its variants to MKP, and test the adaptive problem reduction heuristic in [15] on hard instances of MKP.

The paper is organised as follows. We first describe the problem reduction method introduced in [15] in Section 2. The rationale for applying valid inequalities is also discussed. The GLCI and some variants are described in the framework of general lifting principles [19] in Section 3. A Two-level Core problem Heuristic is proposed in Section 4 to tackle large MKP instances. Computational results are presented in Section 5. The conclusion is given in Section 6.

2. Problem reduction heuristic strengthened by valid inequalities

In the core problem reduction heuristic, an efficiency measure e is employed to rearrange the items into the order $(i_1, i_2, \dots, i_n)$, so that

$$e(i_k) \ge e(i_{k+1}) \tag{2}$$

The items with higher efficiency values are regarded to be more likely included into the knapsacks, and the items with lower efficiency values are regarded to be more likely excluded from the knapsacks. An interval $[a_e, b_e]$ is therefore determined so that x_{i_k} is fixed to 1 if $i_k \in F_e^1 = \{i_k \mid 0 < k < a_e\}$, and x_{i_k} is fixed to 0 if $i_k \in F_e^0 = \{i_k \mid n \ge k > b_e\}$. The remaining undecided items $C_e = \{i_k \mid a_e \le k \le b_e\}$ form the reduced problem defined as

$$\begin{aligned} \text{MKPC} \quad \max \quad & z = \sum_{j \in C_e} c_j x_j + \tilde{z} \\ \text{s.t.} \quad & \sum_{j \in C_e} a_{ij} x_j \le \tilde{b}_i, \quad i = 1, \dots, m \\ & x_j \in \{0, 1\}, \quad j \in C_e \end{aligned}$$

with $\tilde{z} = \sum_{j \in F_e^1} c_j$ and $\tilde{b}_i = b_i - \sum_{j \in F_e^1} a_{ij}$, $i = 1, \dots, m$.

C_e with the smallest cardinality that leads to the optimal solution to the original MKP by solving the reduced MKPC is called the core, and the reduced MKPC is called the core problem accordingly [14].

The exact identification of the core problem requires solving the MKP to optimality. In practice only an approximate core is calculated to include hopefully the actual unknown core with high probability. The core size is a crucial parameter in most core problem heuristics, which is used to balance the accuracy of the approximate core and the computational effort required to solve the core problem. It is typically just a predefined constant in the core based heuristics [20]. Some empirical rules are also suggested in the literature which normally only depends on the number of items [14].

Recently a novel core based heuristic [15] is proposed which is based on the Lagrangian relaxation (LR) method. The Lagrangian relaxation of MKP can be written as

$$LR(\lambda) = \max\{(c - \lambda^T A)^T x + \lambda^T b : x \in \{0, 1\}^n\}$$

where $\lambda \in \mathcal{R}_+^m$ are the Lagrangian multipliers associated with the relaxed knapsack constraints. $LR(\lambda)$ provides an upper bound for the MKP problem, and can be further strengthened by solving the Lagrangian dual problem

$$LD = \min_{\lambda} LR(\lambda) \tag{3}$$

which is a non-smooth convex optimisation problem, and can be solved by the subgradient algorithm [21].

Let $\bar{\lambda}$ be the optimal multipliers to LD. The modified profit of each item in $LR(\bar{\lambda})$ is

$$r_i = c_i - \bar{\lambda}^T a_i, \quad i = 1, \dots, n$$

where a_i is the i-th column of A. The set of optimal solutions of $LR(\bar{\lambda})$ is

$$S = \{x \in \{0, 1\}^n : (2x_i - 1) r_i \ge 0, i = 1, \dots, n\}$$

Based on the observation that x_i tends to be 1 in the optimal solution of MKP if the modified profit r_i takes large positive values, while it tends to be 0 if it takes large negative values, the efficiency measure is chosen as

$$e(i) = r_i, \quad i = 1, \dots, n \tag{4}$$

The approximate core is identified as follows. Let $r_{max} = \max\{|r_i|; i = 1, \dots, n\}$. Given $\epsilon \in \mathcal{R}_+$, the core interval is defined as

$$a_e(\epsilon) = \max\{k \mid r_{i_k} \ge \epsilon r_{max}\} + 1, \quad b_e(\epsilon) = \min\{k \mid -r_{i_k} \ge \epsilon r_{max}\} - 1 \tag{5}$$

Therefore the set of variables fixed to 1 is

$$F_e^1(\epsilon) = \{i_k \mid 0 < k < a_e(\epsilon)\} = \{k \mid r_k \ge \epsilon r_{max}\}, \tag{6}$$

the set of core variables is

$$C_e(\epsilon) = \{i_k \mid a_e \le k \le b_e(\epsilon)\} = \{i; |r_i| < \epsilon r_{max}\}, \tag{7}$$

and the set of variables fixed to 0 is

$$F_e^0(\epsilon) = \{i_k \mid n \ge k > b_e\} = \{k \mid -r_k \ge \epsilon r_{max}\} \tag{8}$$

One unique feature of this core identification approach is that the core size is not pre-determined and can dynamically adapt to the characteristics of each instance. This Dynamically reduced Core Heuristic (DCH) has been comprehensively tested [15] on problems featuring varied coefficient correlation structures and constraint slackness levels. It was found that, by setting $\epsilon = 0.15$, DCH compared well with other problem reduction heuristics in terms of solution quality and estimated core problem sizes, and showed robust effectiveness as problem difficulty increased.

It is well-known [22] that the set of optimal multipliers of LD coincides with the set of optimal solutions of the dual of the LP relaxation of MKP:

$$(\text{MKP} - \text{LP}) \quad \bar{z} = \max\{c^T x; Ax \le b, x \in [0, 1]^n\} \tag{9}$$

Accordingly, the efficiency measure (4) is just the reduced cost of each item, which can be efficiently calculated even for large problems using LP solvers.

Since the LP relaxation can be weak for hard problems, ϵ may have to be large which results in a large core. Intuitively, if the LP can be strengthened by valid inequalities, the identification of the core may be more accurate. Here we will consider the ideal case where the polytope of MKP-LP, $P = \{x \in [0, 1]^n; Ax \le b\}$, is the complete description of the convex hull of the MKP polytope $P_I = Conv\ \{x \in \{0, 1\}^n; Ax \le b\}$.