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Some Applications of Semitensor Product of Matrices in Algebra

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Abstract—A review on semitensor product (STP) of matrices is given. It is a generalization of the conventional matrix product for the case when the dimensions of the factor matrices do not satisfy the requirement. Using it, we investigate some structure-related properties of algebras. First, we consider when an algebra is a Lie algebra. The result reveals the topological structure of all finite-dimensional Lie algebras as the variety of a set of polynomial equations. Then we investigate the invertibility of algebras. Invertibility condition is expressed via STP. Finally, the tensor product of algebras is investigated.

Keywords—Semitensor product of matrices, (Lie) Algebra, Structure matrix.

1. INTRODUCTION

Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$, where $M_{s\times t}$ stands for the set of $s \times t$ real matrices. It is well known that the conventional matrix product AB is well defined as long as the dimension matching condition, $n = p$, is satisfied. When $n \neq p$, a new matrix product, called the semitensor product (STP) of matrices, has been proposed and investigated by the author first in [1]. Unlike Kronecker product, the STP is a generalization of the conventional matrix product.

The main purpose for introducing this new matrix product is to treat higher-dimensional data, i.e., data with indexes of multiplicity greater than two. Using it for multivariable polynomials, it is conventional in dealing with nonlinear mappings.

Recently, in some statistical papers, three-dimensional data have been arranged as a cubic matrix and the corresponding new "matrix" product has been defined and investigated, see, e.g., [2,3] as initial works, and there are several followups. One sees easily that it is in general not convenient in use. First, the product can be defined in different ways and the formulas are complicated. Second, it cannot be generated to the higher-dimensional case. We need a new matrix product which can treat higher-dimensional data easily.

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Another motivation for introducing this new matrix product is from computer science. In computer storage the higher-dimensional data are stored in memory as a long queue, and the program, say in C-language, can find the hierarchies of data by using pointer, pointer-to-pointer, pointer-to-pointer-to-pointer, and so on. Roughly speaking, the STP is designed in such a way that it can search pointers automatically. So, it is convenient in dealing with higher-dimensional data.

When a nonlinear problem is considered, one way to handle the problem is to approximate it by polynomials, say, its Taylor expansion. In fact, a k^{th}-order homogeneous polynomial can be considered as a multilinear mapping over k^{th}-dimensional data. Through such a conversion, STP can naturally be used to deal with nonlinear problems.

Some advantages of STP are listed as follows.

- As a generalization of the conventional matrix product, it inherits almost all the major properties of the conventional matrix product.
- Since the conventional matrix product is a special case of STP, which is associative, it is convenient in manipulating data. (In fact, STP can be expressed as mixed conventional and Kronecker products. But since associativity between these products fails, simplifying or manipulating a mixed expression is extremely difficult.)
- When a vector product is considered, e.g., the cross product in $\mathbb{R}^3$, it can be easily expressed as a STP, while the conventional matrix product can hardly be used.

Recently, the STP of matrices has been used in dealing with several different problems, mostly, control problems. For instance, Cheng [1] used it to convert Morgan's problem (i.e., input-output decoupling problem) to the existence of a nonzero solution of a set of algebraic equations. The nonregular feedback linearization of nonlinear systems was discussed in [4], and some useful formulas and their deductions are based on STP. It has also been used in estimating the boundary of the region of attraction of a stable equilibrium of a power system [5,6]. The STP has also some other applications to group theory and differential geometry [7], to physics [8], etc.

The purpose of this paper is to use it to investigate the structures and properties of algebras. For a finite-dimensional algebra, we assign a structure matrix to it. Then the properties of an algebra can be studied via its structure matrix. A motivation for doing this is: the definition of an algebraic structure can only be used to check whether a set with certain operator(s) meets the definition. But through investigating the structure matrix, we may find all such sets. For instance, we may answer how many three-dimensional Lie algebras there are in the world.

The rest of this paper is organized as follows: Section 2 provides necessary preliminaries for STP of matrices. Section 3 gives the matrix expression of general algebras. A general matrix condition of Lie algebras is presented in Section 4. Then the set of n-dimensional Lie algebras is described as the variety of a set of polynomials. Section 5 considers the invertibility of an algebra. Section 6 considers the tensor product of algebras. Section 7 contains some concluding remarks.

2. SEMITENSOR PRODUCT OF MATRICES

This section gives a brief review on STP of matrices. It plays a fundamental rule in the following discussion. We restrict it to the definitions and some basic properties, which are useful in the sequel. We refer to [4,9] for more details.

DEFINITION 2.1.

1. *Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then we split X into p equal-size blocks as $X^1, \dots, X^p$, which are $1 \times n$ rows. Define the*