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On the asymptotic behavior of the solutions of autonomous equations without unstable invariant manifolds

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We investigate strongly continuous semigroups $\{T(t)\}_{t\geq 0}$ on Banach space X by means of discrete time methods applied to a discrete counterpart $\{T(1)^n\}_{n\geq 0}$. This kind of approach is not new and can be traced back at least to Henry's monograph (see Henry (1981) [13]). The semigroup $\{T(t)\}_{t\geq 0}$ is denoted as hyperbolic, if the usual exponential dichotomy conditions are satisfied, i.e. in particular if invertibility is given on the unstable subspace. A weaker version (without assuming the $T(t)$-invariance of the unstable subspace or even more the invertibility of the operators $T(t)$ on the unstable subspace) is denoted as exponential dichotomy. The latter approach is due to Aulbach and Kalkbrenner dealing with difference equations since in Aulbach and Kalkbrenner (2001) [21] it is clearly indicated that this dichotomy notion lacks L^1-robustness. We show that admissibility properties of the sequence spaces $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$ are sufficient for an exponential dichotomy of $\{T(t)\}_{t\geq 0}$ (see Theorem 3.1). Also if we assume the $T(t)$-invariance of the unstable subspace we show that the above admissibility condition implies the invertibility of the operators $T(t)$ on the unstable subspace (see Theorem 3.2). Thus, hyperbolicity of $\{T(t)\}_{t\geq 0}$ turns out to be equivalent to admissibility of the pair $(\ell^p(\mathbb{N}, X), \ell^q(\mathbb{N}, X))$ (see Corollary 3.1).

1. Introduction

It is a classic topic by now to study the connection between the asymptotic behavior of the solutions of the homogeneous equation $\dot{x} = Ax$ and the existence of bounded solutions for the inhomogeneous equation $\dot{x} = Ax + f$, wherever the inhomogeneity is also bounded. Here A is in general an unbounded linear operator acting on a Banach space X.

Important contributions to this topic were done by Perron [1], Massera and Schäffer [2], Daleckij and Krein [3], Chicone and Latushkin [4], Engel and Nagel [5], Coppel [6], Pazy [7], van Neerven [8], van Minh et al. [9], and Palmer [10].

For the case of discrete-time systems analogous results were first obtained in 1934 by Ta Li [11]. Results in this direction can be found in the works due to Coffman and Schäffer [12], Henry [13], Ben-Artzi and Gohberg [14], Pinto [15], La Salle [16], Pituk [17], and Przyluski and Rolewicz [18].

As it is known, dichotomy means the existence of a bounded projection, P, such that the solutions that start in $Im(P)$ decay to zero and the solutions that start in $Im(I-P)$ are unbounded. If the state space is finite dimensional and $PT(t) = T(t)P$ then the operators $T(t)$ will be automatically invertible on $Im(I-P)$ and the dichotomy concept becomes hyperbolicity (i.e. trajectories decay to zero as $t \to \infty$ and also as $t \to -\infty$). Roughly speaking, we call this hyperbolicity because the phase portrait will look like a hyperbola. However, for infinite-dimensional state-spaces the operators $T(t)$ can fail to be onto on $Im(I-P)$ for dichotomous C_0-semigroup $T(t)_{t\geq 0}$ with $PT(t) = T(t)P$. We will deal with the special case

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where $Im(I-P)$ is not $T(t)$-invariant and we will point out admissibility-type conditions that provide the existence of an exponential dichotomy among the trajectories $T(\cdot)x,\ x\in X$. We will also prove that if we assume the invariance (i.e. $PT(t)=T(t)P$) then the admissibility of the pair $(\ell^p(\mathbb{N},X),\ell^q(\mathbb{N},X))$ implies the invertibility of the operators $T(t)$ on $Im(I-P)$, and therefore the semigroup $\{T(t)\}_{t\ge 0}$ will be hyperbolic.

2. Preliminaries

We use the symbol $\mathbb{R}_+$ to denote the set $\{t\in\mathbb{R}:t\ge 0\}$ and the symbol $\mathbb{N}$ to denote the set of nonnegative integers. Let $\mathbb{N}^*=\mathbb{N}-\{0\}$. Also X will denote a Banach space. By $B(X)$ we denote the Banach algebra of all bounded linear operators acting on the Banach space X, and by $\|\cdot\|$ the norms of vectors and operators on X. As usual, we put

$$\ell^p(\mathbb{N},X)=\left\{x:\mathbb{N}\to X:\sum_{n=0}^{\infty}\|x(n)\|^p<\infty\right\},\quad p\in[1,\infty);$$

$$\ell^\infty(\mathbb{N},X)=\left\{x:\mathbb{N}\to X:\sup_{n\in\mathbb{N}}\|x(n)\|<\infty\right\}.$$

We note that $\ell^p(\mathbb{N},X),\ \ell^\infty(\mathbb{N},X)$ are Banach spaces endowed with the respective norms

$$\|x\|_p=\left(\sum_{n=0}^{\infty}\|x(n)\|^p\right)^{1/p};\qquad \|x\|_\infty=\sup_{n\in\mathbb{N}}\|x(n)\|.$$

Recall now that a family $\mathbf{T}=\{T(t)\}_{t\ge 0}$ of bounded linear operators on a Banach space X is called a *strongly continuous semigroup* if $T(0)=I,\ T(t+s)=T(t)T(s)$ for all $t,s\in\mathbb{R}_+$, and $\lim_{t\downarrow 0}T(t)x=x$, for all $x\in X$. If $\{T(t)\}_{t\ge 0}$ is a strongly continuous semigroup, then there exist constants $\omega\ge 0,\ M\ge 1$ such that $\|T(t)\|\le Me^{\omega t}$, for all $t\ge 0$, see for instance [7, Theorem 2.2, p. 4], or alternatively [5, Proposition 5.5, p. 39]. Therefore we can define

$$\omega(\mathbf{T})=\inf\{\alpha\in\mathbb{R}:\ \exists\,\beta\ge 1 \text{ such that } \|T(t)\|\le\beta e^{\alpha t},\ \text{for all } t\ge 0\}.$$

By Gelfand's theorem for the spectral radius $r(T(t))=e^{t\omega(\mathbf{T})}$, (see for instance Proposition 1.2.2 in [8]).

If $\mathbf{T}=\{T(t)\}_{t\ge 0}$ is a strongly continuous semigroup, then we define by $\mathcal{D}$ to be the set of all $x\in X$ such that $\lim_{t\downarrow 0}t^{-1}(T(t)x-x)$ exists. The *infinitesimal generator* of the semigroup $\mathbf{T}=\{T(t)\}_{t\ge 0}$ is the operator A on X, with the domain $D(A)=\mathcal{D}$, given by

$$Ax=\lim_{t\downarrow 0}\frac{T(t)x-x}{t},\quad x\in D(A).$$

The name "infinitesimal generator" is used because we have that

$$Ax=\left.\frac{dT(t)x}{dt}\right|_{t=0},\quad x\in D(A).$$

Remark 2.1. A strongly continuous semigroup $\mathbf{T}=\{T(t)\}_{t\ge 0}$ can be extended to a strongly continuous group if and only if there exists a number $t_0>0$ such that $T(t_0)$ is an invertible operator [see [5, Proposition, p. 80]].

Consider now the abstract Cauchy problem

$$(1.1)\qquad x'=Ax,\quad x(0)=x_0.$$

If A is the infinitesimal generator of a strongly continuous semigroup and $x_0\in X$, then the function $x(t)=T(t)x_0$ is called a *mild* solution of the differential equation (1.1). If $x_0\in D(A)$, then $x(t)=T(t)x_0$ is a *classical*; that is, differentiable, solution of the differential equation.

We recall now that we say that a semigroup is hyperbolic if X can be decomposed as a direct sum of two subspaces (stable and unstable) such that the semigroup is uniformly exponentially stable for positive time on the stable subspace and uniformly exponentially stable for negative time on the unstable subspace. A linear and bounded operator T is said to be *hyperbolic* if

$$\sigma(T)\cap\Gamma=\emptyset,$$

where $\Gamma=\{z\in\mathbb{C}:|z|=1\}$ denotes the unit circle.

An operator $P\in B(X)$ will be called projection if $P^2=P$. The spectral Riesz projection P for a hyperbolic operator T is given by the formula

$$P=\frac{1}{2\pi i}\int_\Gamma(\lambda I-T)^{-1}d\lambda.$$