Computers and Mathematics with Applications 64 (2012) 48–72

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The dome and the ring: Verification of an old mathematical model for the design of a stiffened shell roof

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ARTICLE INFO

Article history:
Received 7 March 2012
Accepted 7 March 2012
Communicated by L. Demkowicz

Keywords:
Shells
Finite elements
Verification and validation

ABSTRACT

We study an old mathematical model, developed before the computer era, for analyzing the strength of a stiffened shell roof. The specific problem considered is a textbook example presented in K. Girkmann: *Flächentragwerke*, 3rd edition, 1954. Here the roof consists of a spherical dome and a stiffening ring of rectangular cross section attached to the edge of the dome. The problem is to compute the resultant force and moment acting at the junction of the dome and the ring. We approach the old model for solving the problem in two different ways. First we carry out a historical study, where we look for possible improvements of the old model while limiting ourselves to manual computations only. We find a variant of the model which, despite being about as simple as the original one, is considerably more accurate in comparison with recent numerical solutions based on FEM and axisymmetric 3D elastic formulation of the problem. The second approach in our study is to carry out an a posteriori error analysis of our refined old model. The analysis is based on variational methods and on the Hypercircle theorem of the linear theory of elasticity. The error analysis confirms, and largely also explains, the observed – rather high – accuracy of the refined old mathematical model.

1. Introduction

The modeling of shell structures, such as shell roofs, is traditionally one of the most challenging tasks of engineering mathematics. Shells are the sensitive "primadonnas" of structures, both from the viewpoint of engineering design and mathematical modeling.

Sixty years ago the design of shell structures was still based largely on parametrized classical solutions and manual computing. From old textbooks like [1,2] one can get a general idea of this rather advanced engineering science before the era of computers. The basis of the manual computational models was the *classical shell theory*, which was well developed already 60 years ago. The classical shell theory reduces the 3D linear elastic laws to 2D equations, so called *shell equations*, along the middle surface of the shell. Shell equations are still partial differential equations, and even worse, with variable coefficients, so they are not solvable by analytic means in general. Under special geometric or symmetry assumptions, however, the shell equations may be reduced further to ordinary differential equations in one space dimension. The old engineering shell theory covers a collection of such special situations. In most of these cases, further simplification of the 1D shell equations is still needed to allow a classical solution in terms of elementary functions.

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.03.008

The engineer faces a 1D shell problem, e.g., when he wants to certify that a dome-shaped shell roof, as designed, carries its own weight. In this paper we travel backwards in time to see how the engineer handled such a problem in the pre-computer era. The shell roof to be considered is taken from a textbook example presented in [1]. (The same example is found also in [3].) The example is named here the *Girkmann problem* according to its original reference. The roof consists here of a thin spherical dome and a stiffening footring connected to it at the meridional angle $\alpha = 40^\circ$. The ring is of rectangular cross section and connected to the dome along its edge. The material of the whole structure is concrete, assumed homogeneous and linearly elastic in the mathematical model. An equilibrium support at the base of the ring is assumed to balance the weight of the structure. In this setting, the computational problem to be solved is specified as: find the values of the horizontal force (R) and moment (M) by which the dome and the ring act on each other at their intersection. Both R and M are reactions to be evaluated per unit length of the junction line. A more detailed description of the Girkmann problem is given in Section 2.

As stated, the Girkmann problem is part of the certification of the strength of the roof: knowing the reaction force and moment acting at the edge of the dome, the engineer can compute further the stresses in the dome according to shell theory. In particular, he can evaluate the maximal bending stress in the vicinity of the junction—the most critical quantity concerning the strength of the roof. Bending stresses are due to the so called *edge effect* that is characteristic to shell deformations near edges or interfaces.

In [1] it is demonstrated how an approximate solution to the Girkmann problem is found manually. First the classical shell theory applied to the spherical dome is simplified to an approximate engineering shell theory. The latter consists of the so called *membrane theory* (M) and *bending theory* (B) for the shell, each valid approximately under specific loading and edge conditions. For the ring the classical engineering *ring theory* (R) is assumed. Upon combining the engineering shell and ring theories and imposing kinematic continuity constraints at the junction, one obtains the traditional simplified model for determining the two unknown quantities R and M. We refer to this classical textbook model here as the M–B–R model. In the end the M–B–R model reduces to a 2×2 linear system for the unknowns, with given algebraic expressions for the coefficients of the system. Using such a model, a trained engineer of the old generation probably needed only a pencil, logarithmic and trigonometric tables, a back of an envelope, and half an hour to solve the problem for a given design.

But how accurate is such a simple model? —We should be able to answer such a question now, assuming that the 'exact' solution obeys the 3D laws of linear elasticity with the given material parameters of the problem. In cylindrical or spherical coordinates, with the rotational symmetry taken into account, the mathematical problem actually reduces to a 2D linear elastic problem on the vertical cross section of the roof. For the engineers of today, now working with a laptop computer and a FEM code, an accurate numerical solution of such a problem should be routine.

A recent test, however, tells a different story. In [4], the Girkmann problem was announced as a benchmark test for the expert users of finite element software products. The participants were asked to solve the problem using their favorite code and to verify that the error in the computed values of R and M was no more than 5%. The results received from 15 respondents were summarized in [5,6]. The desired accuracy was achieved in only 6 of the 15 solutions. In another 6 solutions the error in M exceeded 100% and in one solution, R was about 20 times and M about 500 times too large and even the sign of M was wrong [6].

In a later contribution to the Girkmann benchmark test, different finite element approaches based on open software were tested, and this time quite accurate results were obtained consistently [7]. What then caused the wide scattering of the results in the earlier test remains largely conjectural. In any case, the Girkmann problem challenge succeeds in underlining the importance of *verification* of numerical results even in the context of relatively simple-looking problems. In general, both verification and *validation* (V&V, see [8,9]) of numerical and mathematical models is of growing importance now that more and more complex problems are becoming numerically solvable and engineering curricula no longer cover classical methods in sufficient detail.

But let us return to the question posed above concerning the accuracy of the traditional manual solution to the Girkmann problem. This was the question that actually inspired the first finite element benchmarking on the problem in [10], but so far this original question has remained unresolved. Our aim here is to close the case and give a precise answer. In the V&V terminology, our aim is to carry out the full verification of the classical model when solving the Girkmann problem. The 2D formulation of the problem assumed in [4–7] (originally due to [10]) is considered here as 'exact'.

The outline of the paper is as follows. In Section 2 we give the precise formulation of the Girkmann problem as a 2D linear elastic problem. In Sections 3 and 4 we approach the classical model for solving the problem in two quite different ways. Section 3 is a historical expedition back to the derivation of the model. Our aim is to find out, to what extent it is possible to improve the classical model so as to make it more accurate without sacrificing its simplicity. In Section 4 we focus on the model variant that we find experimentally to be the most accurate one. We attack this model by methods of mathematical error analysis, with the aim to both certify and explain the observed accuracy of the model. Finally, in Section 5 we present the summary and conclusions of our paper, together with some historical remarks.

In what follows we present first an extended introduction that gives a more detailed outline of the contents of Sections 3 and 4.

Study of the old model (Section 3)

In the old literature little or no attention is given to possible variations of the basic M–B–R model as found in textbooks. In the true accuracy test that we perform here, however, the fine tuning of the M–B–R model turns out to have a significant