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Some ordered fixed point results and the property (P)

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ABSTRACT

In 2010, Kadelburg et al. ([7]) by providing an example showed that a contraction in an ordered metric space is not necessarily a contraction (in the classical sense). Thus fixed point results in ordered metric spaces are generalizations of ones in metric spaces in a sense. In this paper, we give some ordered fixed point results for convex contractions and special mappings which satisfy some contraction conditions. Also, we give some results concerning the property (P).

1. Introduction

The notion of metric spaces was introduced in 1906 by Maurice Frechet. Since then, many researchers have exploited this notion to define various concepts, using different views and ideas. One of the important notions is that of ordered metric spaces. We say that $(X, d, \leq)$ is an ordered metric space whenever $\leq$ is an order on X and (X, d) is a metric space. Although these spaces have been considered by many authors recently and even though ordered metric spaces were introduced and studied a few years ago, there are some old works on these spaces. For example, Vandergraft reviewed the Newton method for convex operators on partially ordered spaces in 1967 [1]. Also, Wolk reviewed continuous convergence on partially ordered spaces in 1975 [2]. Later, Sun and Sun started ordered fixed point theory in 1989 [3] and after some years it was continued by Agarwal et al. [4]. Also, Wanka published a paper concerning approximation theory in ordered spaces in 1996 [5]. In 2010, Altun et al. [6] and Kadelburg et al. [7] proved some fixed point and common fixed point theorems on ordered metric spaces by using a cone. In recent years, ordered fixed point theory has been considered by many authors (see, for example, [8–33]).

Rhoades defined the property (P) on metric spaces in his works [34–36]. Denote, as usual, by $F(T)$ the set of fixed points of the mapping $T : X \to X$. We say that a self-map T has the property (P) whenever $F(T) = F(T^n)$ for all $n \geq 1$, that is, it has no periodic points. Note that $F(T) \subseteq F(T^n)$ for all $n \geq 1$. Recently, two interesting papers have appeared on the property (P) [37,38]. More recently, Alghamdi et al. have studied convex contraction and two-sided convex contraction mappings [39], which were introduced by Istratescu [40] in 1982. We use these notions to obtain some results. In this paper, we give some ordered fixed point results for convex contractions and special mappings which satisfy some contraction conditions and are not necessarily continuous. Also, we give some results concerning the property (P).

2. The main results

Let $(X, \leq)$ be a partially ordered set. We define $X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x\}$. Also, we say that a self-map $T : X \to X$ is orbitally continuous at x whenever for each sequence $\{n(i)\}_{i\geq 1}$ with $T^{n(i)}x \to a$ for some $a \in X$, we have

$$T^{n(i)+1}x \to Ta.$$

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Here, $T^{m+1} = T(T^m)$. Finally, we define the orbit of T at x by

$$O(x, \infty) := \{x, Tx, T^2x, \dots, T^nx, \dots\}$$

and we say that T has the strongly comparable property whenever $(T^{n-1}y, T^ny) \in X_{\leq}$ for all $n \geq 1$ and $m \geq 2$, where $y \in F(T^m)$.

Theorem 2.1. *Let $(X, d, \leq)$ be a complete ordered metric space, $\lambda \in (0, 1)$ and T a self-map on X satisfying the condition*

$$\min\{d^2(Tx, Ty), d(x, y)d(Tx, Ty), d^2(y, Ty)\} - \min\{d^2(x, Tx), d(y, Ty)d(x, Ty), d^2(y, Tx)\} \leq \lambda\, d(x, Tx)d(y, Ty)$$

for all $x, y \in X_{\leq}$. If there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^nx_0) \in X_{\leq}$ for all $n \geq 1$ and T is orbitally continuous at x_0, then T has a fixed point. Moreover, if T has the strongly comparable property, then T has the property (P).

Proof. Define $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0-1}$ for some natural number n_0, then $x_n = x_{n_0}$ for all $n \geq n_0$ and x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \geq 1$. Now for each $n \geq 1$, by using the assumption, we can put $x = x_{n-1}$ and $y = x_n$ in the condition. Thus, we obtain

$$\min\{d^2(x_n, x_{n+1}), d(x_{n-1}, x_n)d(x_n, x_{n+1})\} \leq \lambda\, d(x_{n-1}, x_n)d(x_n, x_{n+1}).$$

Since $\lambda < 1$, $\min\{d^2(x_n, x_{n+1}), d(x_{n-1}, x_n)d(x_n, x_{n+1})\} = d^2(x_n, x_{n+1})$. Hence,

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

By continuing this process we obtain $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$ for all $n \geq 1$. Thus, for each natural number k, we have

$$d(x_n, x_{n+k}) \leq \sum_{i=n}^{n+k-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+k-1} \lambda^i d(x_0, x_1) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$. Since T is orbitally continuous, $x_{n+1} = Tx_n \to Tu$. This implies that $Tu = u$. Now, we prove that T has the property (P). Let $n \geq 2$ be given and $v \in F(T^n)$. Since T has the strongly comparable property, we can put $x = T^{n-1}v$ and $y = T^nv$ in the condition. Thus, we obtain

$$\min\{d^2(T^nv, T^{n+1}v), d(T^{n-1}v, T^nv)d(T^nv, T^{n+1}v)\} \leq \lambda\, d(T^{n-1}v, T^nv)d(T^nv, T^{n+1}v).$$

Thus, $\min\{d^2(v, Tv), d(T^{n-1}v, v)d(v, Tv)\} \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$ and so two cases arise.

Case I. $d^2(v, Tv) \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$.

We claim that $d(v, Tv) = 0$. If $d(v, Tv) > 0$, then $d(v, Tv) = d(T^nv, T^{n+1}v) \leq \lambda\, d(T^{n-1}v, T^nv)$. By putting $x = T^{n-2}v$ and $y = T^{n-1}v$ in the condition, we obtain

$$\min\{d^2(T^{n-1}v, T^nv), d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv)\} \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv).$$

Let $d^2(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv)$. If $d(T^{n-1}v, T^nv) = 0$, then $T^{n-1}v = v$ and so $v = T^nv = Tv$. If $d(T^{n-1}v, T^nv) > 0$, then $d(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)$. Now, let

$$d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv).$$

So we should have $d(T^{n-2}v, T^{n-1}v) = 0$ or $d(T^{n-1}v, T^nv) = 0$ (and so $v = Tv$), because if $d(T^{n-2}v, T^{n-1}v) > 0$ and $d(T^{n-1}v, T^nv) > 0$, then we get $\lambda \geq 1$ which is a contradiction. By continuing this process, we obtain

$$d(v, Tv) = d(T^nv, T^{n+1}v) \leq \lambda\, d(T^{n-1}v, T^nv) \leq \lambda^2\, d(T^{n-2}v, T^{n-1}v) \cdots \leq \lambda^n\, d(v, Tv)$$

which leads us to $\lambda \geq 1$ which is a contradiction. Therefore, in this case we have $d(v, Tv) = 0$ and so $Tv = v$.

Case II. $d(T^{n-1}v, v)d(v, Tv) \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$.

In this case we should have $d(T^{n-1}v, v) = 0$ or $d(v, Tv) = 0$ (and so $v = Tv$). In fact, if $d(T^{n-1}v, v) > 0$ and $d(v, Tv) > 0$, then $\lambda \geq 1$ which is a contradiction. Thus, we have the consequence that $F(T^n) \subseteq F(T)$. Therefore, T has the property (P). □

The following example shows that there are nonlinear and discontinuous mappings which satisfy the condition of Theorem 2.1.

Example 2.1. Let $X = [0, \infty)$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $Tx = 0$ whenever $0 \leq x \leq 10$, $Tx = x - 10$ whenever $10 \leq x \leq 11$ and $Tx = 1.1$ whenever $x > 11$. Then, on putting $\lambda = \frac{1}{2}$, T satisfies the condition of Theorem 2.1.