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Application of the exp-function method for solving nonlinear reaction–diffusion equations arising in mathematical biology

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In this work we consider nonlinear reaction–diffusion equations arising in mathematical biology. We use the exp-function method in order to obtain conventional solitons and periodic solutions. The proposed scheme can be applied to a wide class of nonlinear equations.

1. Introduction

We consider nonlinear reaction–diffusion equations with a convection term of the form

$$u_t = [A(u)u_x]_x + B(u)u_x + C(u), \tag{1}$$

where $u = u(x, t)$ is an unknown function, and $A(u), B(u), C(u)$ are arbitrary smooth functions. The indices x and t denote differentiation with respect to these variables. Eq. (1) generalizes a great number of well-known nonlinear second-order evolution equations describing various processes in biology [1–3].

There are particular cases such as the classical Burgers equation

$$u_t = u_{xx} + \lambda_1 u u_x, \tag{2}$$

and the well-known Fisher equation [4]

$$u_t = u_{xx} + \lambda_2 u - \lambda_3 u^2, \tag{3}$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. A particular case of Eq. (1) is the Murray equation [1,2]

$$u_t = u_{xx} + \lambda_1 u u_x + \lambda_2 u - \lambda_3 u^2, \tag{4}$$

which can be considered as a generalization of the Fisher and Burgers equations.

Construction of particular solutions for nonlinear equations of the form (1) remains an important problem. Finding approximate solutions that have a biological interpretation is of fundamental importance. Recently, Cherniha [5] used a constructive method for finding new ansätze and exact solutions.

Very recently, He and Wu [6] proposed a straightforward and concise method, called the exp-function method, for obtaining generalized solitary solutions and periodic solutions; applications of the method can be found in [7–13] for

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solving nonlinear equations. The exp-function method has been used to solve many equations [6–13]; it is also valid for solving difference–differential equations [14,15], and variable-coefficient equations [16,17]. In addition, n-soliton solutions and rational solutions can be constructed by the method [11–13]. All applications verified that the exp-function method is straightforward, concise and effective in obtaining generalized solitary solutions and periodic solutions of nonlinear evolution equations. The main merits of this method over the other methods are that it gives more general solutions with some free parameters. As a consequence of application of the exp-function method we can usually obtain exact solutions which have to be simplified. But sometimes it is not easy to make simplifications taking into account the cumbersome expressions. So, this is the main deficiency of this method. Kudryashov and co-workers [18–20] showed the critical points of the exp-function method and other travelling wave solution methods.

Kudryashov [18] analyzed common errors of the recent papers in which the solitary wave solutions of nonlinear differential equations were presented. He showed that many popular methods used in finding the exact solutions were equivalent each other. He illustrated several cases where authors presented certain functions for describing solutions but did not use arbitrary constants.

Kudryashov [19] considered the Korteweg–de Vries and the Korteweg–de Vries–Burgers equations. He analyzed the paper of Wazzan [21] and demonstrated that all of his solutions were not new and could be transformed to known solutions.

Kudryashov [20] analyzed an application of the exp-function method to search for exact solutions of nonlinear differential equations. Possibilities for using the exp-function method and other approaches in mathematical physics were discussed.

Various methods have some merits and deficiencies with respect to the problem considered and there is no single best method for investigating the exact solutions of the nonintegrable nonlinear equations of the type stated above. What is more, it is likely that the method chosen for solving nonintegrable nonlinear equations should be selected according to the form of nonlinearity of the equation and the pole of its solution as well. Keeping all this in mind, the Murray equation involves both; that is to say, it involves strong nonlinearity and has a pole of second order. That is why the Murray equation is a challenging equation for the application of any method. At this point, it could be thought that the exp-function method may be the right approach to employ for addressing the Murray equation, and it was our starting point, because in the exp-function method, the assumed solution of the problem is in exponential form with many parameters while most of the methods mentioned above give the solutions in series form. This shows the flexibility of the method and if one can determine the parameters accordingly, one may obtain various solutions of the problem considered [10].

The solution procedure of the exp-function method, carried out with the aid of Maple, is of utter simplicity and this method can easily be extended to addressing other kinds of nonlinear equations. We will use the exp-function method for solving this problem.

2. The exp-function method

The exp-function method was first proposed by He and Wu [7] and systematically studied for solving a class of nonlinear partial differential equations. We consider the general nonlinear partial differential equation of the type

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \ldots\right) = 0. \tag{5}$$

Using a transformation

$$\eta = kx + wt \tag{6}$$

where k and w are constants, we can rewrite Eq. (5) as the following nonlinear ODE:

$$Q(u, u', u'', u''', u^{(\mathrm{iv})}, u^{(\mathrm{v})}, \ldots) = 0. \tag{7}$$

According to the exp-function method, as developed by He and Wu [7], we assume that the wave solutions can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} \tag{8}$$

where p, q, d and c are positive integers which are known to be further determined, and a_n and b_m are unknown constants. We can rewrite Eq. (7) in the following equivalent form:

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-\mathrm{d}\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{9}$$

This equivalent formulation plays an important and fundamental part in finding the analytic solution of problems. To determine the values of c and p, we balance the linear term of highest order of Eq. (8) with the highest order nonlinear term. Similarly, to determine the values of d and q, we balance the linear term of lowest order of Eq. (7) with the lowest order nonlinear term.