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Common fixed points of hybrid maps and an application

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ABSTRACT

We introduce a new notion that is a generalization of Definition 2.1, Kamran and Cakić (2008) [3]. Using this notion, we establish a new result, that is, coincidence and fixed points for two hybrid pairs of nonself-maps satisfying an implicit relation. This result generalizes the multivalued version of some known results (see, Imdad and Ali (2007) [12] and the references therein). Also, the same result generalizes Theorem 2.8, Liu et al. (2005) [4]. As application, we prove a coincidence point theorem for hybrid nonself-maps in product spaces.

1. Introduction and preliminaries

Since the past five decades, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Also, the fixed point theory is a beautiful mixture of analysis, topology, and geometry.

Hybrid contraction maps are contractive conditions involving multivalued mappings and single-valued mappings. Hybrid fixed point theory for these mappings is a new development in the domain of contraction-type multivalued theory (see, e.g., [1–7] and the references therein). Second, several authors proved some common fixed point theorems for nonself-mappings (see, for example, [8–11]). Third, some fixed point theorems for mappings satisfying implicit relations have appeared (see, for instance, [12,13]).

Let X be a metric space with metric d. Then, for $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. Let $CB(X)$ denote the class of all nonempty closed bounded subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X. Let H be the generalized Hausdorff metric on $CL(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\},$$

for every $A, B \in CL(X)$. A point $p \in X$ is said to be a *fixed point* of $I : X \to X$ (resp. $S : X \to CL(X)$) if $p = Ip$ (resp. $p \in Sp$). The point $p \in X$ is said to be a *common fixed point* of $I : X \to X$ and $J : X \to X$ (resp. $I : X \to X$ and $S : X \to CL(X)$) if $p = Ip = Jp$ (resp. $p = Ip \in Sp$). $p \in X$ is called a *coincidence point* of $I : X \to X$ and $J : X \to X$ (resp. $I : X \to X$ and $S : X \to CL(X)$) if $Ip = Jp$ (resp. $Ip \in Sp$). It is obvious that any common fixed point is a coincidence point but its converse need not be true.

Definition 1.1 (*[14]*)**.** Maps $I : X \to X$ and $S : X \to CB(X)$ are *weakly compatible* if they commute at their coincidence points, i.e., $ISx = SIx$ whenever $Ix \in Sx$ for some $x \in X$.

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Definition 1.2 (*[1]*)**.** Let $S : X \to CB(X)$. The map $I : X \to X$ is said to be S-*weakly commuting at* $x \in X$ if $IIx \in SIx$.

The weak compatibility leads to the S-weak commutativity at the coincidence point of I and S but its converse need not be true (see, [1]).

Definition 1.3 (*[15]*)**.** Maps $I : X \to X$ and $J : X \to X$ are said to satisfy *property* (E. A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jx_n = t$ for some $t \in X$.

Definition 1.4 (*[4]*)**.** Let $I, J : X \to X$ and $S, T : X \to CB(X)$. The pairs (I, S) and (J, T) are said to satisfy *the common property* (E. A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$, and $A, B \in CB(X)$ such that $\lim_{n\to\infty} Sx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B$.

Following [13], let Ψ be the family of real lower semi-continuous functions $F : [0, \infty)^6 \to \Re$, $\Re :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1) F is non-increasing in the 3rd, 4th, 5th and 6th coordinate variables,
(ψ_2) there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with
(ψ_{21}) $F(u, v, v, u, u + v, 0) \leq 0$ or (ψ_{22})$F(u, v, u, v, 0, u + v) \leq 0$, we have $u \leq hv$, and
(ψ_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

For the sake of completeness, we enlist some examples which are essentially included in [12,13].

Example 1.5. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - h\left[a \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\} + (1 - a)\left[\max\left\{t_2^2, t_3t_4, t_5t_6, \frac{1}{2}t_3t_6, \frac{1}{2}t_4t_5\right\}\right]^{\frac{1}{2}}\right],$$

where $h \in (0, 1)$ and $0 \leq a \leq 1$. One can verify that $F \in \Psi$.

Example 1.6. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - h\left[\max\left\{t_2^2, t_3t_4, t_5t_6, t_4t_6, t_3t_5\right\}\right]^{\frac{1}{2}},$$

where $h \in \left(0, \frac{1}{\sqrt{2}}\right)$. One can show that $F \in \Psi$.

Example 1.7. Define $F : [0, \infty)^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6,$$

where $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$. One can deduce that $F \in \Psi$.

We state the following theorem for convenience.

Theorem 1.8 (*[12, Theorem 3.1]*)**.** *Let S and I be self-mappings of a metric space (X, d) such that*

(i) *S and I satisfy property* (E. A),
(ii) $\forall x, y \in X$ *and* $F \in \Psi$,

$$F(d(Sx, Sy), d(Ix, Iy), d(Ix, Sx), d(Iy, Sy), d(Ix, Sy), d(Iy, Sx)) \leq 0,$$

(iii) *$I(X)$ is a complete subspace of X.*

Then

(a) *the pair (S, I) has a coincidence point,*
(b) *the pair (S, I) has a common fixed point provided that it is weakly compatible.*

The rest of this paper is organized as follows. In the next section, we introduce a new notion (see, Definition 2.5) which is a generalization of Definition 2.1 [3]. Also, we establish a coincidence and fixed point theorem for two hybrid pairs of nonself-maps satisfying an implicit relation. This theorem generalizes the multivalued version of Theorem 1.8. Also, the same theorem generalizes Theorem 2.8 [4]. Finally, in Section 3, we apply the main theorem for obtaining a coincidence point theorem for hybrid nonself-maps in product spaces.