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# On using a modified Legendre-spectral method for solving singular IVPs of Lane–Emden type

H. Adibi *, A.M. Rismani

*Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., 15914, Tehran, Iran*

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ABSTRACT

In this paper, approximate solutions of singular initial value problems (IVPs) of the Lane–Emden type in second-order ordinary differential equations (ODEs) are obtained by an improved Legendre-spectral method. The Legendre–Gauss points are used as collocation nodes and Lagrange interpolation is employed in the Volterra term. The results reveal that the method is effective, simple and accurate.



## 1. Introduction

The Lane–Emden equation describes the gravitational potential of a self-gravitating spherically symmetric polytropic fluid, as well as the equilibrium density distribution in a self-gravitating sphere of polytropic isothermal gas, which is of fundamental importance in the fields of stellar structure [1], radiative cooling, modeling of clusters of galaxies, astrophysics [2], etc. We consider the following Lane–Emden type equation [1,3]:

$$y'' + \frac{2}{t}y' + f(y) = 0, \quad 0 < t \le 1 \tag{1.1}$$

subject to

$$y(0) = a, \qquad y'(0) = 0 \tag{1.2}$$

where $t$ and $y$ denote the independent and dependent variables, respectively, the primes denote differentiation with respect to $t$, $f(y)$ is a nonlinear function of $y$, and $a$ is a constant. It should be noted that the most general form of singular initial value problems of Lane–Emden type is as follows:

$$y'' + \frac{2}{t}y' + f(t, y) = g(t), \quad 0 < t \le 1 \tag{1.3}$$

subject to conditions (1.2), where $f(t, y)$ is a continuous real valued function, and $g(t) \in C[0, 1]$. This has also been handled analytically by using perturbation methods [4], Adomian's decomposition method [5], the quasilinearization method of Bellman and Kalaba [6], the piecewise linearization technique [7] and a variational method [8]. The variational iteration method [2] and the collocation method have been utilized for solving the Lane–Emden equation arising in astrophysics in [2,9]. Also a numerical method can be found in [10].

* Corresponding author. Tel.: +98 21 64542524; fax: +98 66497930.
*E-mail address:* adibih@aut.ac.ir (H. Adibi).

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In this paper, we first transform Eq. (1.1) into a Volterra integral equation and then solve it numerically by a spectral method. Overall, the spectral class of solution methods, based on using orthogonal polynomials, are implemented in various ways, such as using pantograph type delay differential equations [11] and Volterra type integral equations [12]. The spectral method provides the most convenient computer implementations.

The remainder of the paper is organized as follows: In Section 2, we transform Eq. (1.3) into a Volterra integral equation with sufficiently smooth kernel. In Section 3, we present an improved Legendre-collocation method. In Section 4, numerical results for some problems are investigated and the corresponding tables are presented. Finally in Section 5 the report ends with a brief conclusion.

## 2. Volterra's integral equation formulation

Eq. (1.1) can be written as

$$L(y) = ty'' + 2y' = -tf(t, y) + tg(t) \tag{2.1}$$

and the solution of $L(y) = 0$ together with the method of variation of parameters yields

$$y(t) = \frac{1}{t}\left(C + \int_0^t s^2 f(u(s))\mathrm{d}s\right) + D - \int_0^t (sf(s, y(s)) - sg(s))\mathrm{d}s \tag{2.2}$$

where $C$ and $D$ are constants. Imposing the initial conditions of (1.2), we obtain

$$y(t) = a + \int_0^t \left(\frac{s^2}{t} - s\right)(f(s, y(s)) - g(s))\mathrm{d}s \tag{2.3}$$

which is a nonlinear Volterra integral equation of the second kind.

## 3. The Legendre-collocation method

In order to use a spectral method, we consider the collocation points as the set of $N$ Legendre–Gauss, or Gauss–Radua, or Gauss–Lobatto points $\{t_j\}_{j=1}^N$.

If we do so, entering the collocation points, (2.3) gets replaced by

$$y(t_j) = a + \int_0^{t_j} \left(\frac{s^2}{t_j} - s\right)(f(s, y(s)) - sg(s))\mathrm{d}s, \quad t_j \in [-1, 1],\ j = 1, 2, \ldots, N. \tag{3.1}$$

The main difficulty in obtaining a high rate of accuracy is computing the integral term in (3.1). In fact for small values of $t_j$, there is little information available for $y(s)$. To overcome this difficulty, the integral interval $(0, t_j]$ is transferred to the fixed interval $(-1, 1]$. We first make the following simple linear transformation:

$$s(t, \theta) = \frac{t}{2}\theta + \frac{t}{2}, \quad -1 \le \theta \le 1. \tag{3.2}$$

Then (3.1) takes the form

$$y(t_j) = a + \frac{t_j}{2}\int_0^{t_j} \left(\frac{s^2(t_j, \theta)}{t_j} - s(t_j, \theta)\right)\big(f(s(t_j, \theta),\ y(s(t_j, \theta))) - g(s(t_j, \theta))\big)\,\mathrm{d}\theta. \tag{3.3}$$

Using an $N$-point Gauss quadrature rule related to the Legendre weights $\{w_j\}$ in $[-1, 1]$ gives

$$y(t_j) = a + \frac{t_j}{2}\sum_{k=1}^{N} \left(\frac{s^2(t_j, \theta_k)}{t_j} - s(t_j, \theta_k)\right)\big(f(s(t_j, \theta_k), y(s(t_j, \theta_k))) - g(s(t_j, \theta_k))\big)\, w_k, \tag{3.4}$$

where $\{\theta_k\}_{k=1}^N$ coincide with the collocation points $\{t_j\}_{j=1}^N$. We now need to represent $f(s, y(s))$ in terms of $y_k$ for $k = 1, 2, \ldots, N$. To this end, we expand them, using Lagrange interpolation polynomials, in the vector sense as

$$y(s) \approx \sum_{p=1}^{N} y_p\, Ł_p(s), \tag{3.5}$$

where $Ł_p$ is the $p$-th Lagrange basis function and is expressed in terms of Legendre functions by

$$Ł_p(s) = \sum_{k=1}^{N} \beta_k, pP_k(s). \tag{3.6}$$