Computers and Mathematics with Applications 60 (2010) 2153–2160

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A power of an entire function sharing one value with its derivative

Ji-Long Zhang [a,*], Lian-Zhong Yang [b]

[a] Beihang University, LMIB and School of Mathematics & Systems Science, Beijing, 100191, PR China

[b] Shandong University, School of Mathematics, Jinan, Shandong, 250100, PR China

ARTICLE INFO

Article history:
Received 16 May 2009
Received in revised form 2 August 2010
Accepted 2 August 2010

Keywords:
Uniqueness theorems
Shared value
Entire function

ABSTRACT

In this paper, we investigate uniqueness problems of entire functions that share one value with one of their derivatives. Let f be a non-constant entire function, n and k be positive integers. If f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$, then $f^n = (f^n)^{(k)}$, and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c is a non-zero constant and $\lambda^k = 1$. This result shows that a conjecture given by Brück is true when $F = f^n$, where $n \geq 2$ is an integer.

1. Introduction

In what follows, a meromorphic (resp. entire) function always means a function which is meromorphic (resp. analytic) in the whole complex plane. We will use the standard notation in Nevanlinna's value distribution theory of meromorphic functions; see, e.g. [1].

We say that two meromorphic functions f and g share $a \in \mathbb{C}$ IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities). Let m and p be positive integers. We denote by $N_p\left(r, 1/(f-a)\right)$ the counting function of the zeros of $f-a$ where m-fold zeros are counted m times if $m \leq p$ and p times if $m > p$.

Recently, a widely studied subtopic of the uniqueness theory has been the consideration of shared value problems relative to a meromorphic function F and its kth derivative $F^{(k)}$. In order to get the uniqueness of sharing one value of F and $F^{(k)}$, some deficient assumption is needed. The reader is invited to see the recent papers [2–7].

The purpose of this paper is to study a power of an entire function sharing one value with its derivative. We will give some results concerning Brück's Conjecture, which is mentioned later.

Let f be a non-constant entire function and n be a positive integer. If f^n and $(f^n)'$ share 1 CM, then there exists an entire function α such that

$$\frac{(f^n)'-1}{f^n-1} = e^{\alpha}.$$

Rewriting above equation, we have

$$g_1 + g_2 + g_3 = 1, \tag{1.1}$$

where $g_1 = (f^n)'$, $g_2 = -e^{\alpha}f^n$, $g_3 = e^{\alpha}$.

There are many results on a combination of three meromorphic functions

$$f_1 + f_2 + f_3 = 1 \tag{1.2}$$

in uniqueness theory. The following result is a useful one. As for the proof; see, e.g. [8].

* Corresponding author.
E-mail addresses: jilong_zhang@mail.sdu.edu.cn (J.-L. Zhang), lzyang@sdu.edu.cn (L.-Z. Yang).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.08.001

Theorem 1.1. *Let f_j $(j = 1, 2, 3)$ be meromorphic functions satisfying (1.2). If f_1 is not a constant, and*

$$\sum_{j=1}^{3} N_2(r, 1/f_j) + \sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \quad r \in I,$$

where $\lambda < 1$, $T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\}$, I denotes a set of $r \in (0, \infty)$ with infinite linear measure. Then either $f_2 = 1$ or $f_3 = 1$.

Applying Theorem 1.1 on (1.1), the present authors [8] got:

Theorem 1.2. *Let f be a non-constant entire function, $n \geq 7$ be an integer. If f^n and $(f^n)'$ share 1 CM and, then $f^n = (f^n)'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a non-zero constant.

It is natural to ask whether n can be reduced in Theorem 1.2. In fact, there are much more relations between g_j in (1.1) than f_j in (1.2). By studying this, we give a result improving Theorem 1.2 in Section 2. In Section 3, we consider a power of an entire function sharing 1 IM with its derivative. We provide some concluding remarks in Section 4.

2. Sharing 1 CM

In order to get a general result, we consider f^n sharing 1 CM with its kth derivative, where k is a positive integer, and obtain the following result:

Theorem 2.1. *Let f be a non-constant entire function, n and k be positive integers. If f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k + 1$, then $f^n = (f^n)^{(k)}$, and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z}, \tag{2.1}$$

where c is a non-zero constant and $\lambda^k = 1$.

In order to prove Theorem 2.1, we need the following lemma.

Lemma 2.2 ([1, Lemma 4.3]). *Let f be a non-constant meromorphic function and $P(f)$ be a polynomial in f with constant coefficients. Let b_j $(j = 1, \ldots, q)$ be distinct finite values. If $q > \deg P$, then*

$$m\left(r, \frac{P(f)f'}{(f - b_1)(f - b_2)\cdots(f - b_q)}\right) = S(r, f).$$

We begin to prove Theorem 2.1:

Proof. Denote

$$F = f^n. \tag{2.2}$$

Since F and $F^{(k)}$ share 1 CM, then there exists an entire function α, such that

$$F^{(k)} - 1 = e^{\alpha}(F - 1). \tag{2.3}$$

Suppose first that e^{α} is a non-constant entire function. By differentiation, we have

$$F^{(k+1)} = \alpha' e^{\alpha}(F - 1) + e^{\alpha}F'. \tag{2.4}$$

Combining (2.3) with (2.4) yields

$$F^{(k+1)}F - \alpha' F^{(k)}F - F^{(k)}F' = F^{(k+1)} - \alpha'(F^{(k)} + F) - F' + \alpha'. \tag{2.5}$$

By induction, we deduce from (2.2) that

$$F^{(k)} = \sum_{\lambda} c_{\lambda} f^{l_0^{\lambda}} (f')^{l_1^{\lambda}} \cdots \left(f^{(k)}\right)^{l_k^{\lambda}}, \tag{2.6}$$

where $l_0^{\lambda}, \ldots, l_k^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k} l_j^{\lambda} = n$, $n - k \leq l_0^{\lambda} \leq n - 1$ and c_{λ} are constants.