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On a stochastic logistic equation with impulsive perturbations

Meng Liu [a,b,*], Ke Wang [b]

[a] School of Mathematical Science, Huaiyin Normal University, Huaian 223300, PR China

[b] Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China

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ABSTRACT

A stochastic logistic model with impulsive perturbations is proposed and investigated. First, we give a new definition of a solution of an impulsive stochastic differential equation (ISDE), which is more convenient for use than the existing one. Using this definition, we show that our model has a global and positive solution and obtain its explicit expression. Then we establish the sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solution. The critical value between weak persistence and extinction is obtained. In addition, the limit of the average in time of the sample path of the solution is estimated by two constants. Afterwards, the lower-growth rate and the upper-growth rate of the solution are estimated. Finally, sufficient conditions for global attractivity are established.

1. Introduction

The investigation of logistic equation has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its importance. Since population dynamics in the real world is inevitably affected by environmental noise which is an important component in an ecosystem, several authors (see e.g. [1–7]) have investigated the following stochastic logistic equation

$$dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), \tag{1}$$

where $x(t)$ is the population size and $B(t)$ is a standard Brownian motion. Many important results of solutions of Eq. (1) have been obtained.

On the other hand, the theory of impulsive differential equation appears as a natural description of several kinds of real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Processes of this type are often studied in various fields of science and technology: population dynamics, ecology, biological systems, physics, pharmacokinetics, optimal control, etc.; see e.g., the monographs [8,9]. Various population dynamical systems of impulsive differential equations have been proposed and investigated extensively. Many important and interesting results on the dynamical behaviors for such systems have been found; see e.g., [10–17] and the references therein. Recently, stability of stochastic differential equation (SDE) with impulsive effects has been done by Sakthivel and Luo [18], Zhao et al. [19], Li and Sun [20], Li et al. [21] and Li et al. [22]. However, so far as our knowledge is concerned, very little amount of work on the stochastic population dynamics with impulsive effects has been done.

In this paper, we will study the following stochastic logistic system with impulsive perturbations:

$$\begin{cases} dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), & t \neq t_k,\ k \in N \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N \end{cases} \tag{2}$$

* Corresponding author at: Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China.
E-mail address: liumeng0557@sina.com (M. Liu).

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where N denotes the set of positive integers, $0 < t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$, $r(t)$, $a(t)$ and $\sigma(t)$ are continuous bounded functions on $R_+ := [0, +\infty)$. The following additional restrictions on (2) are natural for biological meanings:

$$\inf_{t\in R_+} a(t) > 0, \qquad 1 + b_k > 0, \quad k \in N.$$

When $b_k > 0$, the perturbation stands for planting of the species, while $b_k < 0$ stands for harvesting. The main aims of this work are to investigate how impulses affect on the existence of positive solutions, permanence, persistence, extinction and global attractivity of Eq. (2). Our results show that the impulse does not affect all of these properties if the impulsive perturbations are bounded. However, if the impulsive perturbations are unbounded, some properties could be changed significantly. The important contributions of this paper is therefore clear.

To proceed, we need some appropriate definitions of persistence. Based on these definitions, we shall establish the persistence and extinction results for Eq. (2). Ma and his co-workers proposed the concepts of weak persistence [23], non-persistence in the mean [24] and persistence in the mean [24] for some deterministic models. Especially, Wang and Ma [25] pointed out the fact that there is only threshold between weak persistence and extinction of populations for general non-autonomous population models.

Definition 1.
- $x(t)$ is said to be extinctive if $\lim_{t\to+\infty} x(t) = 0$.
- $x(t)$ is said to be nonpersistent in the mean if $\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = 0$.
- $x(t)$ is said to be weakly persistent if $\limsup_{t\to+\infty} x(t) > 0$.
- $x(t)$ is said to be persistent in the mean if $\liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds > 0$.
- $x(t)$ is said to be stochastically permanent if for every $\varepsilon \in (0, 1)$, there are constants $\beta > 0, \delta > 0$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1 - \varepsilon, \qquad \liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \delta\} \ge 1 - \varepsilon.$$

The rest of the paper is arranged as follows. In Section 2, we give a new definition of solution of ISDE, which is more convenient for use than the existing definition. Then we show that Eq. (2) has a global and positive solution for any positive initial condition and give its explicit expression. In Section 3, sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the population represented by Eq. (2) are established. The critical value between weak persistence and extinction is obtained. Moreover, the limit of the average in time of the sample path of the solution is estimated by two constants. In Section 4, the lower-growth rate and the upper-growth rate of the solutions are estimated. In Section 5, we investigate the global attractivity of Eq. (2). In the last section, we give the conclusions and illustrate our main results through some examples and figures.

2. Global positive solution

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions. Let $B(t)$ denote a standard Brownian motion defined on this probability space. Moreover, we always assume that a product equals unity if the number of factors is zero.

Definition 2. Consider the following ISDE:

$$\begin{cases} dX(t) = F(t, X(t))dt + G(t, X(t))dB(t), & t \ne t_k,\ k \in N \\ X(t_k^+) - X(t_k) = B_k X(t_k), & k \in N \end{cases} \tag{3}$$

with initial condition $X(0)$. A stochastic process $X(t) = (X_1(t), \dots, X_n(t))^T$, $t \in R_+$, is said to be a solution of ISDE (3) if

(i) $X(t)$ is $\mathcal{F}_t$-adapted and is continuous on $(0, t_1)$ and each interval $(t_k, t_{k+1}) \subset R_+,\ k \in N$; $F(t, X(t)) \in \mathcal{L}^1(R_+; R^n)$, $G(t, X(t)) \in \mathcal{L}^2(R_+; R^n)$, where $\mathcal{L}^k(R_+; R^n)$ is all R^n valued measurable $\{\mathcal{F}_t\}$-adapted processes $f(t)$ satisfying $\int_0^T |f(t)|^k dt < \infty$ a.s. (almost surely) for every $T > 0$;

(ii) for each $t_k,\ k \in N$, $X(t_k^+) = \lim_{t\to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist and $X(t_k) = X(t_k^-)$ with probability one;

(iii) for almost all $t \in [0, t_1]$, $X(t)$ obeys the integral equation

$$X(t) = X(0) + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dB(s). \tag{4}$$

And for almost all $t \in (t_k, t_{k+1}],\ k \in N$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k}^t F(s, X(s))ds + \int_{t_k}^t G(s, X(s))dB(s). \tag{5}$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k,\ k \in N$ with probability one.

Remark 1. One of the definitions of the solution of an ISDE is as follows.