Computers and Mathematics with Applications 58 (2009) 1169–1179

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Properties of certain analytic multivalent functions defined by a linear operator

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ARTICLE INFO

Article history:
Received 23 December 2007
Received in revised form 5 October 2008
Accepted 16 October 2008

Keywords:
Analytic function
Multivalent function
Convolution
Linear operator
Integral operator
Subordination

ABSTRACT

Let $A(p,k)(p,k\in N=\{1,2,3,\ldots\})$ be the class of functions $f(z)=z^p+a_{p+k}z^{p+k}+\cdots$ which are analytic in the unit disk $E=\{z:|z|<1\}$. By using a linear operator $L_{p,k}(a,c)$, we introduce a new subclass $T_{p,k}(a,c,\delta;h)$ of $A(p,k)$ and derive some interesting properties for the class $T_{p,k}(a,c,\delta;h)$.

1. Introduction and preliminaries

Let $A(p,k)(p,k\in N=\{1,2,3,\ldots\})$ be the class of functions of the form

$$f(z)=z^p+\sum_{m=k}^{\infty}a_{p+m}z^{p+m} \tag{1.1}$$

which are analytic in the unit disk $E=\{z:|z|<1\}$. We denote $A(p,1)=A_p$, $A_1=A$. Also, we denote by K and $S^*(\alpha)$ the usual subclasses of A whose members are convex and starlike of order α, $0\le\alpha<1$, in E, respectively. The class $A(p,k)$ is closed under the Hadamard product (or convolution)

$$f(z)*g(z)\equiv(f*g)(z)=z^p+\sum_{m=k}^{\infty}a_{p+m}b_{p+m}z^{p+m}=(g*f)(z)\quad(z\in E),$$

where

$$f(z)=z^p+\sum_{m=k}^{\infty}a_{p+m}z^{p+m},\qquad g(z)=z^p+\sum_{m=k}^{\infty}b_{p+m}z^{p+m}.$$

Let the function $\varphi_{p,k}(a,c)$ be defined by

$$\varphi_{p,k}(a,c;z)=z^p+\sum_{m=k}^{\infty}\frac{(a)_m}{(c)_m}z^{p+m}\quad(z\in E), \tag{1.2}$$

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2008.10.100

where $c \neq 0, -1, -2, \ldots, (\lambda)_0 = 1$ and $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$ for $m \in N$. Carlson and Shaffer [1] defined a convolution operator on A by

$$L(a, c)f(z) = \varphi_{1,1}(a, c) * f(z) \quad (f(z) \in A). \tag{1.3}$$

Similarly, we define a linear operator $L_{p,k}(a, c)$ on $A(p, k)$ by

$$L_{p,k}(a, c)f(z) = \varphi_{p,k}(a, c) * f(z) \quad (f(z) \in A(p, k)). \tag{1.4}$$

It is easily seen from (1.2) and (1.4) that

$$z(L_{p,k}(a, c)f(z))' = aL_{p,k}(a+1, c)f(z) - (a-p)L_{p,k}(a, c)f(z). \tag{1.5}$$

Clearly $L_{p,k}(a, c)$ maps $A(p, k)$ into itself and $L_{p,k}(c, c)$ is identity. If $a \neq 0, -1, -2, \ldots$, then $L_{p,k}(a, c)$ has an inverse $L_{p,k}(c, a)$. Note also that

$$L_{p,k}(p+1, p)f(z) = zf'(z)/p.$$

For a real number $\lambda > -p$ and a function $f(z) \in A(p, k)$, we define the generalized Libera integral operator $J_{p,\lambda}$ (see [2]) by

$$J_{p,\lambda}f(z) = \frac{\lambda+p}{z^{\lambda}} \int_0^z t^{\lambda-1} f(t)\mathrm{d}t \tag{1.6}$$

and the generalized Ruscheweyh derivative $D^{\lambda+p-1}$ (see [3]) by

$$D^{\lambda+p-1}f(z) = f(z) * \frac{z^p}{(1-z)^{\lambda+p}}. \tag{1.7}$$

It can be easily verified that

$$L_{p,k}(\lambda+p, \lambda+p+1)f(z) = J_{p,\lambda}f(z) \tag{1.8}$$

and that

$$L_{p,k}(\lambda+p, 1)f(z) = D^{\lambda+p-1}f(z). \tag{1.9}$$

Also, we write $L_{p,1}(a, c) = L_p(a, c)$, $L_1(a, c) = L(a, c)$ and $\varphi_{p,1}(a, c) = \varphi_p(a, c)$.

Let $f(z)$ and $g(z)$ be analytic in E. We say that the function $f(z)$ is subordinate to $g(z)$ in E, and we write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in E such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in E$. If $g(z)$ is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

Throughout our present investigation, we assume that $p, k \in N$, $a > 0$, $c \neq 0, -1, -2, \ldots$, $\delta \geq 0$ and $h(z)$ is analytic and convex univalent in E with $h(0) = 1$.

By using the operator $L_{p,k}(a, c)$, we now introduce and investigate the following subclass of $A(p, k)$.

Definition. A function $f(z) \in A(p, k)$ is said to be in the class $T_{p,k}(a, c, \delta; h)$ if and only if

$$(1-\delta)\frac{L_{p,k}(a, c)f(z)}{z^p} + \delta\frac{L_{p,k}(a+1, c)f(z)}{z^p} \prec h(z) \quad (z \in E). \tag{1.10}$$

Also, we write $T_{p,1}(a, c, \delta; h) = T_p(a, c, \delta; h)$ and $T_1(a, c, \delta; h) = T(a, c, \delta; h)$.

Remark 1. In a recent paper, Yang and Liu [4] introduced a subclass $H(p, k, \lambda, \delta, A, B)$ of $A(p, k)$ and satisfied the following subordination condition:

$$(1-\delta)\frac{D^{\lambda+p-1}f(z)}{z^p} + \delta\frac{D^{\lambda+p}f(z)}{z^p} \prec \frac{1+Az}{1+Bz},$$

where $\delta > 0$, $\lambda > -p$, $-1 \leq B < A \leq 1$.

It is easy to see that, if we set $a = \lambda + p$, $c = 1$ and $h(z) = \frac{1+Az}{1+Bz}$ in the class $T_{p,k}(a, c, \delta; h)$, then it reduces to the class $H(p, k, \lambda, \delta, A, B)$.

In this paper, we aim at proving such results as inclusion relationships and convolution properties for the class $T_{p,k}(a, c, \delta; h)$. The results presented here would provide extensions of those given in a number of earlier works (see Yang and Liu [4], Aouf [5,6], Obradovic [7], Patel and Rout [8], Saitoh [9] and others).