Computers and Mathematics with Applications 57 (2009) 1187–1195

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Discrete schemes for Gaussian curvature and their convergence☆

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ARTICLE INFO

Article history:
Received 22 April 2008
Received in revised form 26 December 2008
Accepted 9 January 2009

Keywords:
Discrete surfaces
Gaussian curvature
Discrete curvature
Geometric modelling
Angular defect schemes

ABSTRACT

The popular angular defect schemes for Gaussian curvature only converge at the regular vertex with valence 6. In this paper, we present a new discrete scheme for Gaussian curvature, which converges at the regular vertex with valence greater than 4. We show that it is impossible to build a discrete scheme for Gaussian curvature which converges at the regular vertex with valence 4 by a counterexample. We also study the convergence property of other discrete schemes for Gaussian curvature and compare their asymptotic errors by numerical experiments.

1. Introduction

Estimation of intrinsic geometric invariants is important in a number of applications such as in computer vision, computer graphics, geometric modelling and computer aided design. It is well known that Gaussian curvature is one of the most essential geometric invariants for surfaces. However, in the classical differential geometry, this invariant is well defined only for C^2 smooth surfaces. In the field of modern-computer-related geometry, one often uses C^0 continuous discrete triangular meshes to represent smooth surfaces approximately. Hence, estimation of accurately Gaussian curvature for triangular meshes is demanded strongly.

In the past years, a wealth of different methods for estimating Gaussian curvature have been proposed in the vast literature of applied geometry. These methods can be divided into two classes. The first class is for computing Gaussian curvature based on the local fitting or interpolation technique [1–5], while the second class is for giving discretization formulas which represent the information about the Gaussian curvature [6–9]. In this paper, our focus is on the methods in the second class and our main aim is to present a new discretization scheme for Gaussian curvature which has better convergence property than the previous discretization schemes.

Let M be a triangulation of the smooth surface S in $\mathbb{R}^3$. For a vertex $\mathbf{p}$ of M, suppose that $\{\mathbf{p}_i\}_{i=1}^n$ is the set of the one-ring neighbor vertexes of $\mathbf{p}$. The set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}(i = 1, \ldots, n)$ of n Euclidean triangles forms a piecewise linear approximation of S around $\mathbf{p}$. Throughout the paper, we use the following conventions $\mathbf{p}_{n+1} = \mathbf{p}_1$ and $\mathbf{p}_0 = \mathbf{p}_n$. Let γ_i denote the angle $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and let the angular defect at $\mathbf{p}$ be $2\pi - \sum_i \gamma_i$.

A popular discretization scheme for computing Gaussian curvature is in the form of $(2\pi - \sum_i \gamma_i)/E$, where E is a geometric quantity. In general, one takes E as $A(\mathbf{p})/3$ and obtains the following approximation

$$G^{(1)} := \frac{3(2\pi - \sum\limits_i \gamma_i)}{A(\mathbf{p})}, \tag{1}$$

☆ Zhiqiang Xu is supported by the NSFC grant 10871196. Guoliang Xu is supported by NSFC grant 60773165 and National Key Basic Research Project of China (2004CB318000).

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doi:10.1016/j.camwa.2009.01.024

where $A(\mathbf{p})$ is the sum of the areas of triangles $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$. In [6], another scheme

$$G^{(2)} := \frac{2\pi - \sum_i \gamma_i}{S_p} \tag{2}$$

is given, where

$$S_p := \sum_i \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right]$$

is called the module of the mesh at $\mathbf{p}$. In [10,11], the discretization approximation $G^{(1)}$ is modified as

$$G^{(3)} := \frac{2\pi - \sum_i \gamma_i}{\frac{1}{2}\sum_i \mathrm{area}(\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}) - \frac{1}{8}\sum_i \cot(\gamma_i)d_i^2}, \tag{3}$$

where d_i is the length of edge $\mathbf{p}_i\mathbf{p}_{i+1}$. There are different viewpoints for explaining the reason why the angular defect closely relates to the Gaussian curvature, including the viewpoints of the Gaussian–Bonnet theorem, Gaussian map and Legendre's formula (see the next section in details).

Asymptotic analysis for the discretization schemes have been given in [4,6,9]. In [4], the authors show that the discretization scheme $G^{(1)}$ is not always convergent to the true Gaussian curvature for the non-uniform data. In [6], the authors prove that the angular defect is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures and show that the scheme $G^{(2)}$ converges to the exact Gaussian curvature in a linear rate provided $\mathbf{p}$ is a regular vertex with valence six. Moreover, in [6], the authors show that 4 is the only value of the valence such that the angular defect depends upon the principal directions. In [9], Xu proves that the discretization scheme $G^{(1)}$ has a quadratic convergence rate if the mesh satisfies the so-called parallelogram criterion, which requires valence 6. Therefore, one hopes to construct a discretization scheme which converges over any discrete mesh. But in [12], Xu et al. show that it is impossible to construct a discrete scheme which is convergent over any discrete mesh. Hence, we have to be content with the discretization schemes which converge under some conditions. According to past experiences [6,12,13], we regard a discretization scheme as desirable if it has the following properties:

(1) It converges at regular vertexes, at least for sufficiently large valence (the definition of the regular vertex will be given in Section 2);
(2) It converges at umbilical points, i.e., the points satisfying $k_m = k_M$ where k_m and k_M are two principal curvatures.

As stated before, the previous discretization schemes, including $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$, only converge at the regular vertex with valence 6. In [6], a method for computing the Gaussian curvature at the regular vertex with valence unequal to 4 is described. But the method requires two meshes with valences n_1 and n_2 ($n_1 \neq 4, n_2 \neq 4, n_1 \neq n_2$). In this paper, we will construct a discretization scheme which converges at the regular vertex with valence not less than 5, and also at umbilical points with any valence. Moreover, the discretization scheme requires only a single mesh. Hence, the new scheme is more desirable. Furthermore, we show that it is impossible to construct a discretization scheme which is convergent at the regular vertex with valence 4. Therefore, the convergence problem remains open for the regular vertexes with valence 3. Here, it should be noted that the pointwise convergence discussed in this paper is different from the convergence in norm as discussed in [14,15].

The rest of the paper is organized as follows. Section 2 describes some notations and definitions and Section 3 shows three viewpoints for expressing the relation between the angular defect and Gaussian curvature. In Section 4, we study the convergence property of a modified discretization Gaussian curvature scheme. We present in Section 5 a new discretization scheme and prove that the scheme has a good convergence property, which is the central result of the paper. In Section 6, for the regular vertex with valence 4, we show that it is impossible to build a discretization scheme which is convergent to the real Gaussian curvature. Some numerical results are given in Section 7.

2. Preliminaries

In this section, we introduce some notations and definitions used throughout the paper (see also Fig. 1). Let S be a given smooth surface and $\mathbf{p}$ be a point over S. Suppose that the set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}$, $i = 1, \ldots, n$, of n Euclidean triangles forms a piecewise linear approximation of S around $\mathbf{p}$. The vector from $\mathbf{p}$ to $\mathbf{p}_i$ is denoted as $\overrightarrow{\mathbf{p}\mathbf{p}_i}$. The normal vector and tangent plane of S at the point $\mathbf{p}$ are denoted by $\mathbf{n}$ and Π, respectively. We denote the projection of $\mathbf{p}_i$ onto Π as $\mathbf{q}_i$, and define the plane containing $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{p}_i$ as Π_i. Then we let κ_i denote the curvature of the plane curve $S \cap \Pi_i$ at $\mathbf{p}$. The distances from $\mathbf{p}$ to $\mathbf{p}_i$ and $\mathbf{q}_i$ are denoted as η_i and l_i, respectively. Let γ_i and β_i denote $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and $\angle\mathbf{q}_i\mathbf{p}\mathbf{q}_{i+1}$. The two principal curvatures at $\mathbf{p}$ are denoted as k_m and k_M. Let $\eta = \max_i \eta_i$. The following results are presented in [6,9,13]:

$$\frac{l_i}{\eta_i} = 1 + O(\eta), \qquad \beta_i = \gamma_i + O(\eta^2), \tag{4}$$