Computers and Mathematics with Applications 63 (2012) 14–24

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An application of H differentiability to generalized complementarity problems over symmetric cones☆

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ARTICLE INFO

Article history:
Received 26 April 2011
Received in revised form 15 October 2011
Accepted 17 October 2011

Keywords:
Generalized complementarity problems
Symmetric cone
Euclidean Jordan algebra
C-function
H-differentiability
Strong semismoothness

ABSTRACT

In this paper, we focus on a generalized complementarity problems over symmetric cone GSCCP(f, g) when the underlying functions f and g are H-differentiable. By introducing the concepts of relatively uniform Cartesian P-property, relatively Cartesian $P(P_0)$-property, the Cartesian semimonotone (E_0)-property (strictly Cartesian semimonotone (E)-property), and the relatively regular point with respect to the merit function $\Psi(x)$, we extend various similar results proved in GCP(f, g) to generalized complementarity problems over symmetric cone GSCCP(f, g) and establish various conditions on f and g to get a solution to GSCCP(f, g).

1. Introduction

Consider generalized complementarity problems over symmetric cone, denoted by GSCCP(f, g), which is to find $x \in J$ such that

$$f(x) \in K, \; g(x) \in K, \quad \langle f(x), g(x)\rangle = 0, \tag{1.1}$$

where f and g are continuous mappings from J into itself, $K \subset J$ is a symmetric cone, J is an n-dimensional Euclidean space and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

From the structure theorems of a Euclidean Jordan algebra, we know that for a given Euclidean Jordan algebra J and the corresponding symmetric cone K, we have

$$J = J_1 \times J_2 \times \cdots \times J_m \qquad \text{and} \qquad K = K_1 \times K_2 \times \cdots \times K_m,$$

where each n_v-dimensional space J_v is a simple Jordan algebra (which is not the direct sum of two Euclidean Jordan algebras) with the corresponding symmetric cone K_v, and $\sum_{v=1}^m n_v = n$. Moreover, for $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in J with $x_v, y_v \in J_v$, one also has

$$x \circ y = (x_1 \circ y_1, x_2 \circ y_2, \ldots, x_m \circ y_m)^T$$

and

$$\langle x, y\rangle = \sum_{v=1}^{m} \langle x_v, y_v\rangle.$$

☆ The project is supported by National Natural Science Foundation of China grant 11071041.

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doi:10.1016/j.camwa.2011.10.046

Therefore, the generalized complementarity problem over symmetric cone (1.1) is equivalent to

$$f_\upsilon(x) \in K_\upsilon,\ g_\upsilon(x) \in K_\upsilon, \quad \langle f_\upsilon(x), g_\upsilon(x)\rangle = 0, \tag{1.2}$$

for each $\upsilon \in \{1, 2, \ldots, m\}$.

This problem has a wide range of applications Refs. [1,2]. It is closely related to the optimality conditions for symmetric cone linear programming (SCLP). Moreover, problem (1.1) includes semidefinite complementarity problems (SDCP), second-order cone complementarity problems (SOCCP), and linear and nonlinear complementarity problems (LCP/NCP) as special cases. For example, when $K = R^n_+$, symmetric cone complementarity problems reduce to the nonlinear complementarity problems (NCPs).

The concepts of and H-differentials were introduced in Ref. [3] to study the injectivity of nonsmooth functions. It has been shown in Ref. [3] that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitzian function Refs. [4], the Bouligand subdifferential of a semismooth function Refs. [5,6], and the C-differential of a C-differentiable function Ref. [7] are examples of H-differentials. Moreover, the H-differentiable function need not be locally Lipschitzian nor directionally differentiable. These concepts give useful and unified treatments for many problems when the underlying functions are not necessarily locally Lipschitzian nor semismooth, see Refs. [8–12].

In this paper, we focus on a generalized complementarity problems over symmetric cone GSCCP(f, g) when the underlying functions f and g are H-differentiable. Also, we introduce the concepts of relatively uniform Cartesian P-property, relatively Cartesian $P(P_0)$-property, the Cartesian semimonotone (E_0)-property (strictly Cartesian semimonotone (E)-property), and the relatively regular point with respect to the merit function $\Psi(x)$ to establish some conditions on f and g to get a solution for generalized complementarity problems over symmetric cone GSCCP(f, g). Furthermore, we show that our results extend various similar results proved for the nonlinear generalized complementarity problems GCP(f, g) in Ref. [11].

The paper is organized as follows. In Section 2, we introduce some useful mathematical results on Euclidean Jordan algebras associated with symmetric cone. In Section 3, we describe the H-differentiable of the c-function and its merit function over symmetric cone. The corresponding important properties of this complementarity function are discussed then. In Section 4, we give stationary points analysis of merit function with an H-differential, and impose different conditions on the functions f and g to get a solution for GSCCP(f, g).

In our notation, ^{T} denotes transpose; int K denotes the interior of K; $a \succeq b$ or $a \succ b$ means that $a - b \in K$ or $a - b \in \operatorname{int} K$, respectively. We denote $[\cdot]_+/[\cdot]_- : J \to K/-K$ as the nearest point projection onto $K/-K$, i.e., $[x]_+ := \operatorname{argmin}\{\|x-y\| \mid y \in K\}$($[x]_- := \operatorname{argmin}\{\|x-y\| \mid y \in -K\}$), respectively.

2. Preliminaries

In this section, we give some basic results of Euclidean Jordan algebras, which is a basic tool extensively used in this paper. Our presentation is concise and without proofs. For more details, the reader is referred to Ref. [13].

$(J, \circ)$ is called a Jordan algebra if a bilinear mapping $J \times J \to J$ denoted by "$\circ$" is defined for any $x, y \in J$ such that

(i) $x \circ y = y \circ x$,
(ii) $L_x L_{x^2} = L_{x^2} L_x$,

where $x^2 = x \circ x$, L_x is a linear transformation of J defined by $L_x(y) = x \circ y$. Note that Jordan algebras are not necessarily associative, i.e., $x \circ (y \circ z) \neq (x \circ y) \circ z$ in general.

A Jordan algebra J is called Euclidean if an associative inner product "$\langle\cdot,\cdot\rangle$" is defined, i.e., $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ holds for every $x, y, z \in J$. A Jordan algebra has an identity, if there exists a (necessarily unique) element e such that $x \circ e = e \circ x = x$ for all $x \in J$. Throughout this paper, we assume that J is a Euclidean Jordan algebra with an identity element e.

An element $c \in J$ is called idempotent if $c \circ c = c$. Idempotents c and d are orthogonal if $c \circ d = 0$. An idempotent c is primitive if c cannot be expressed by the sum of two other nonzero idempotents. We denote the maximum possible number of primitive orthogonal idempotents by r, which is called the rank of J. The rank of J is in general different from the dimension of J. A set of idempotents $\{c_1, c_2, \ldots, c_r\}$ is called a Jordan frame if they are orthogonal to each other, i.e. $c_i \circ c_j = 0$ for any $i, j \in \{1, \ldots, r\}$ with $i \neq j$, and $c_1 + \cdots + c_r = e$.

The set $\{x^2 : x \in J\}$ is called the cone of squares of Euclidean Jordan algebra$(J, \circ, \langle\cdot,\cdot\rangle)$. The following theorem can be found in Ref. [13].

Theorem 2.1. *A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra.*

Theorem 2.2 (*Spectral Decomposition Theorem*)**.** *Let J be a Euclidean Jordan algebra with rank r. Then, for every $x \in J$, there exists a Jordan frame $c_1, c_2, \ldots, c_r$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $x = \sum_{i=1}^r \lambda_i c_i$. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ (with their multiplicities) are uniquely determined by x.*

The real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are called the eigenvalues of x, which are continuous functions with respect to x. The trace of x is defined by $\sum_{i=1}^r \lambda_i$, denoted as $\operatorname{Tr}(x)$, which is a linear function with respect to x. The determinant of x is defined