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Majorization problem for certain class of p-valently analytic function defined by generalized fractional differintegral operator

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ABSTRACT

In this paper we investigate a majorization problem for a subclass of p-valently analytic function involving a generalized fractional differintegral operator. Some useful consequences of the main result are mentioned and relevance with some of the earlier results are also pointed out.

1. Introduction

Let functions f and g are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}. \tag{1.1}$$

We say that f is majorized by g in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

if there exist a function $\varphi(z)$, analytic in $\mathbb{U}$ such that

$$|\varphi(z)| \le 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

Note that majorization is closely related to quasi-subordination [1].

Further, we say that the function f is subordinated to g, and write $f(z) \prec g(z)$, $z \in \mathbb{U}$, if there exists a function w analytic in $\mathbb{U}$, with

$$|w(z)| < 1 \quad \text{and} \quad w(0) = 0 \quad (z \in \mathbb{U}), \tag{1.4}$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.5}$$

In particular, if $f(z)$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We recall here the following generalized fractional integral and generalized fractional derivative operators due to Srivastava et al. [2] (see also [3]).

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Definition 1. For real numbers $\lambda > 0$, μ and η, Saigo hypergeometrc fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1-\frac{t}{z}\right) f(t)\, dt, \tag{1.6}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}) \quad (z \to 0; \varepsilon > \max\{0, \mu-\eta\} - 1),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 2. Under the hypotheses of Definition 1, Saigo hypergeometric fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \left\{ z^{\lambda-\mu} \displaystyle\int_0^z (z-t)^{-\lambda} {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z}\right) f(t)\, dt \right\} & (0 \leq \lambda < 1); \\ \dfrac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\eta} f(z) \quad (n \leq \lambda < n+1; n \in \mathbb{N}), & \end{cases} \tag{1.7}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

It may be remarked that

$$I_{0,z}^{\lambda,-\lambda,\eta} f(z) = D_z^{-\lambda} f(z) \quad (\lambda > 0)$$

and

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1),$$

where $D_z^{-\lambda}$ denotes fractional integral operator and D_z^{λ} denotes fractional derivative operator considered by Owa [4].

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1.8}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}$. Recently Goyal and Prajapat [5], introduced generalized fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} : \mathcal{A}_p \to \mathcal{A}_p$, by

$$\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) & (0 \leq \lambda < \eta+p+1, z \in \mathbb{U}); \\ \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z) & (-\infty < \lambda < 0, z \in \mathbb{U}). \end{cases} \tag{1.9}$$

It is easily seen from (1.9) that for a function f of the form (1.8), we have

$$\begin{aligned} \mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\eta-\mu)_n}{(1+p-\mu)_n (1+p+\eta-\lambda)_n} a_{p+n} z^{p+n} \\ &= z^p {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z) \\ &\quad (z \in \mathbb{U};\ p \in \mathbb{N};\ \mu, \eta \in \mathbb{R};\ \mu < p+1; -\infty < \lambda < \eta+p+1) \end{aligned} \tag{1.10}$$

where $*$ denotes usual Hadamard product of analytic functions and ${}_pF_q$ is well known generalized hypergeometric function.

The operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$ satisfies the following three-term recurrence relation:

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))^{(j+1)} = (p+\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z))^{(j)} - (\eta+j-\lambda)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))^{(j)}. \tag{1.11}$$

Note that

$$\mathcal{S}_{0,z}^{0,0,0} f(z) = f(z), \qquad \mathcal{S}_{0,z}^{1,1,1} f(z) = \mathcal{S}_{0,z}^{1,0,0} f(z) = \frac{zf'(z)}{p}$$

and

$$\mathcal{S}_{0,z}^{2,1,1} f(z) = \frac{zf'(z) + z^2 f''(z)}{p^2}.$$

We also note that

$$\mathcal{S}_{0,z}^{\lambda,\lambda,\eta} f(z) = \mathcal{S}_{0,z}^{\lambda,\mu,0} f(z) = \Omega_z^{\lambda,p} f(z),$$