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Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces

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ABSTRACT

In this paper, we establish coupled coincidence and common coupled fixed point theorems for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces. Presented theorems extend, generalize and improve many existing results in the literature. An example is given.

1. Previous definitions and results

The fixed point theorems in partially ordered metric spaces play a major role to prove the existence and uniqueness of solutions for some differential and integral equations. Thus, the attraction of fixed point theorems to a large number of mathematicians is understandable.

One of the most interesting fixed point theorems in ordered metric spaces was investigated by Ran and Reurings [1]. Ran and Reurings [1] applied their result to linear and nonlinear matrix equations. Then, many authors obtained several interesting results in ordered metric spaces (see [2–19]).

Bhaskar and Lakshmikantham [20] initiated the study of a coupled fixed point in ordered metric spaces and applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. For more works in coupled and coincidence point theorems, we refer the reader to [21–28].

Some authors generalized the concept of metric spaces in different ways. Mustafa and Sims [29] introduced the notion of G-metric space in which the real number is assigned to every triplet of an arbitrary set as a generalization of the notion of metric spaces. Based on the notion of G-metric spaces, Mustafa et al. [30–33] obtained some fixed point theorems for mappings satisfying various contractive conditions. Abbas and Rhoades [34] initiated the study of common fixed point in G-metric spaces, while Saadati et al. [35] studied some fixed point theorems in partially ordered G-metric spaces. For more results in G-metric spaces, we refer the reader to [36–39].

In 2010, Abbas et al. [40] introduced the concepts of w and w^*-compatible mappings. Abbas et al. [41] utilized the concept of w and w^*-compatibility to prove an interesting uniqueness theorem of coupled fixed point in G-metric spaces. For more results of coupled fixed point in G-metric spaces, see [42–44].

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In the sequel, the letters $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all natural numbers, respectively. Consistent of Mustafa and Sims [29], the following definitions and results will be needed in the sequel.

Definition 1.1 ([29]). Let X be a nonempty set, and let $G: X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically, a *G-metric* on X. The pair (X, G) is called *G-metric space*.

Definition 1.2 ([29]). Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the *limit* of the sequence (x_n), if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$. We say that the sequence (x_n) is G-convergent to x, or (x_n) G-converges to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\varepsilon > 0$ there exists a natural number k, such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition 1.1 ([29]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *(x_n) is G-convergent to x,*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*
(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.3 ([29]). Let (X, G) be a G-metric space. A sequence (x_n) is called a *G-Cauchy sequence*, if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq k$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.2 ([29]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence (x_n) is G-Cauchy,*
(2) *for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq k$.*

Proposition 1.3 ([29]). *Let (X, G) be a G-metric space. Then, $f: X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 1.4 ([29]). *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.4 ([29]). A G-metric space (X, G) is called *G-complete* if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.5 ([43]). Let (X, G) be a G-metric space. A mapping $F: X \times X \to X$ is said to be *continuous* if for any two G-convergent sequences (x_n) and (y_n) converging to x and y respectively, $(F(x_n, y_n))$ is G-convergent to $F(x, y)$.

The following definition was introduced by Bhaskar and Lakshmikantham in [20].

Definition 1.6 ([20]). Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$. Then the map F is said to have *mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is,

$$x_1 \leq x_2 \quad \text{implies } F(x_1, y) \leq F(x_2, y) \text{ for all } y \in X$$

and

$$y_1 \leq y_2 \quad \text{implies } F(x, y_2) \leq F(x, y_1) \text{ for all } x \in X.$$

Inspired by Definition 1.6, Lakshmikantham and Ćirić [21] introduced the concept of a g-mixed monotone mapping.

Definition 1.7 ([21]). Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$. Then, the map F is said to have *mixed g-monotone property* if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y, that is,

$$gx_1 \leq gx_2 \quad \text{implies } F(x_1, y) \leq F(x_2, y) \text{ for all } y \in X$$

and

$$gy_1 \leq gy_2 \quad \text{implies } F(x, y_2) \leq F(x, y_1) \text{ for all } x \in X.$$