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An upper bound for the distance between a zero and a critical point of a solution of a second order linear differential equation

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ABSTRACT

This paper presents an upper bound for the distance between a zero and a critical point of a solution of the second order linear differential equation $(p(x)y')' + q(x)y(x) = 0$, with $p(x), q(x) > 0$. It also compares it with previous results.

1. Introduction

The qualitative analysis of the distribution of the zeroes of the solutions of the equation

$$(p(x)y')' + q(x)y = 0, \quad x > x_0, \tag{1}$$

has become a discipline in itself since the early results of Sturm in [1]. Since then many papers have tried to provide more insights on the problem and powerful tools have been devised for that.

In this sense one of the most popular problems has been the calculation of bounds for the distance between consecutive zeroes and, associated to it, the calculation of the distance between a zero and the critical point (i.e. a zero of the derivative of a solution of (1)) immediately before or afterwards. In fact, the calculation of lower bounds has provided inequalities as famous as that of Lyapunov and La Vallée–Poussin, both of which have generated in turn a lot of subsequent literature (basically improvements and extensions), partially collected in the excellent monography of [2].

The tools used there have been very varied: real analysis [3–5], Prufer transformations [6], Opial inequality [7] or extensions of Boyd's and Brink's work on the Sobolev inequality [8].

However for the calculation of upper bounds between zeroes the amount of research published so far is significantly lower and older, and typically linked to the estimation of the number of zeroes of a solution of (1) on a given interval [6,9].

If one focuses his search on the determination of upper bounds between zeroes and critical points the situation becomes even worse, up to the point it is almost impossible to find a single reference that treats specifically the topic, which is somewhat bizarre in a way given the success that the equivalent problem with lower bounds has enjoyed. The only possibility left to the interested reader seems to be to reuse the methods employed in the calculation of upper bounds between zeroes with the hope that they can be applicable to this problem. [6] is an example of that.

With this landscape in mind, the purpose of this paper is to present a method to find such upper bounds between zeroes and critical points of solutions of (1), by making smart use of the properties of the functions appearing in a Prufer

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transformation associated to that equation. The problem of determining such upper bounds is relevant not only for the qualitative analysis of the solutions of (1) but also for methods which use them to set bounds for the value of the solutions of (1), like those presented in [10,11] and [12] (in fact the nature of the expressions obtained in [12] makes that concrete method especially sensitive to poor upper bounds for the mentioned distances, an issue that the present method can dramatically improve). In addition to the upper bounds, some consequences will also be derived for special cases of the functions $p(x), q(x)$.

The organization of the paper is as follows. Section 2 will prove the main results. Section 3 will provide some examples to illustrate advantages and disadvantages of the method presented here versus the existing literature. Finally Section 4 will draw several conclusions.

2. Main results

The following extended mean value theorem for integrals, which condenses theorems 3 and 4 of [11], is the key for the results of this paper:

Theorem 1 (*Extended Mean Value Theorem for Integrals*)**.** *Let $a(x), f(x), g(x), m(x)$ be piecewise continuous functions on $[a, b]$ with $f(x), m(x) \geq 0$. Let K be defined by*

$$K = \frac{\int_a^b a(x)f(x)dx}{\int_a^b f(x)dx}. \tag{2}$$

Then if $\frac{f(x)}{m(x)}$ is monotonic increasing on $[a, b]$ one has

$$\frac{\int_a^b \min\{a(s);\ x \leq s \leq b\}m(x)dx}{\int_a^b m(x)dx} \leq K \leq \frac{\int_a^b \max\{a(s);\ x \leq s \leq b\}m(x)dx}{\int_a^b m(x)dx}. \tag{3}$$

If $\frac{f(x)}{m(x)}$ is monotonic decreasing on $[a, b]$ one has

$$\frac{\int_a^b \min\{a(s);\ a \leq s \leq x\}m(x)dx}{\int_a^b m(x)dx} \leq K \leq \frac{\int_a^b \max\{a(s);\ a \leq s \leq x\}m(x)dx}{\int_a^b m(x)dx}. \tag{4}$$

With the aid of Theorem 1 it is possible to prove the next theorem:

Theorem 2. *Let $y(x)$ be a solution of (1) with $p(x)$ and $q(x)$ piecewise continuous and positive on an interval $I \in \Re$. Let a, b be real numbers such that $[a, b] \in I$. Suppose that $a(x)$ is any positive piecewise continuous function on $[a, b]$.*

Then, if $y'(a) = y(b) = 0$ one has

$$\int_a^b \min\left\{\frac{1}{p(s)a(s)}, x \leq s \leq b\right\} a(x)dx \int_a^b \min\left\{\frac{q(s)}{a(s)}, a \leq s \leq x\right\} a(x)dx \leq \frac{\pi^2}{4}, \tag{5}$$

and

$$\int_a^b \max\left\{\frac{1}{p(s)a(s)}, x \leq s \leq b\right\} a(x)dx \int_a^b \max\left\{\frac{q(s)}{a(s)}, a \leq s \leq x\right\} a(x)dx \geq \frac{\pi^2}{4}. \tag{6}$$

If $y(a) = y'(b) = 0$ one has

$$\int_a^b \min\left\{\frac{1}{p(s)a(s)}, a \leq s \leq x\right\} a(x)dx \int_a^b \min\left\{\frac{q(s)}{a(s)}, x \leq s \leq b\right\} a(x)dx \leq \frac{\pi^2}{4}, \tag{7}$$

and

$$\int_a^b \max\left\{\frac{1}{p(s)a(s)}, a \leq s \leq x\right\} a(x)dx \int_a^b \max\left\{\frac{q(s)}{a(s)}, x \leq s \leq b\right\} a(x)dx \geq \frac{\pi^2}{4}. \tag{8}$$

Proof. Let us focus first on proving (5). To that end, let us define the following Prufer transformation of the Eq. (1)

$$\tan \Phi(x) = \frac{\sin \Phi(x)}{\cos \Phi(x)} = \frac{v(x)}{y(x)}, \tag{9}$$