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Triangular and skew-symmetric splitting method for numerical solutions of Markov chains ☆

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ABSTRACT

In this paper, a theorem is presented to indicate that there exists a nonnegative constant $\epsilon \geq 0$ such that the matrix $A = Q^{\mathrm{T}} + \epsilon I$ is a positive-definite matrix, where $I \in \mathbb{R}^{n\times n}$ is an identity matrix and $Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ is a matrix with positive diagonal and nonpositive off-diagonal elements. Then a class of triangular and skew-symmetric splitting (TSS) iteration method is applied to solve the positive-definite linear system $Ax = b$ for obtaining the stationary probability vector of an irreducible Markov chain. Theoretical analyses show that the TSS iteration method converges unconditionally to the unique solution of the linear system, with the upper bound of its contraction factor dependent only on the spectrum of the triangular part and independent of the eigenvectors of the matrices involved. Moreover, the inexact triangular and skew-symmetric splitting (ITSS) iteration method, which employs certain Krylov subspace methods as the inner iteration processes at each step of the outer TSS iteration method, is proposed to accelerate the convergence of the TSS iteration method. Numerical experiments are used to illustrate the effectiveness of the TSS and ITSS iteration methods.

1. Introduction

Markov chains are one of the most important kinds of models in simulation. The use of Markov chains is of interest in a wide range of applications. Such examples are the information retrieval and web ranking [1–4], queueing systems [5–8], stochastic automata networks [9–11], manufacturing systems and inventory control [12,13] and communication systems [14–16]. For analyzing their performance measures, it is required to find their stationary probability distributions π by solving the linear system

$$\pi Q = 0, \quad \pi > 0, \qquad \pi e = 1, \tag{1}$$

where Q is an $n \times n$ generator whose elements q_{ij} denote the rate of transition of the chain from state i to state j and $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$.

It is well known that there exists a unique stationary probability distribution π whose elements are strictly greater than zero for a finite irreducible and aperiodic Markov chains; see, e.g., [17–20]. Hence, the numerical solutions to the linear system (1) is feasible. For simplicity, we rewrite (1) as the following homogeneous linear system

$$\bar{A}x = 0, \quad \text{with } \bar{A} = Q^{\mathrm{T}}, x = \pi^{\mathrm{T}}, \tag{2}$$

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where $\bar{A}$ and x are the transposes of the generator matrix Q and π, respectively. In particular, the coefficient matrix $\bar{A}$ has zero column sum, positive diagonal entries and non-positive off-diagonal entries.

Recently, there is a large amount of work devoted to solving the linear system (2). Such examples are the direct methods in [21–24,20], additive and multiplicative Schwarz methods (with overlap) in [25–28], the well-known Krylov subspace methods in [29–33] and some preconditioning techniques in [34,35,9,6,36,37,23,38–40]. Furthermore, multigrid methods based on aggregation of Markov states have been studied in the literature [41–46]. Along the lines of multigrid methods for general linear systems [47], all these multilevel methods can be regarded as the direct or indirect extensions of two-level iterative aggregation/disaggregation methods, which have been used and analyzed for Markov chains in [48–51]. More numerical methods like hybrid algorithms can be found in [52,53].

In this paper, we will explore the applications of the triangular and skew-symmetric splitting (TSS) iteration method for computing the stationary probability vector of the linear system (2), where the coefficient matrix $\bar{A}$ which has the one-dimensional null space is singular. We believe that this is the first time that Markov chain problems are analyzed by using the TSS iteration method in this context. Our main goal is to present the TSS iteration method as one more possible tool for the numerical solutions of Markov chains.

However, as far as we know, the TSS iteration method has been developed and discussed for solving positive-definite linear systems; see, e.g., [54–57]. Hence, the TSS iteration method cannot be directly applied to calculate the linear system (2). In Section 2, we consider a regularized linear system as given below and present a theorem to indicate that there exists a nonnegative constant $\epsilon \geq 0$ such that the matrix $A = \bar{A} + \epsilon I = Q^{\mathrm{T}} + \epsilon I$ is a positive-definite matrix. In Section 3, we apply the TSS iteration method to solve the regularized linear system and discuss its convergence. In Section 4, we analyze the choice of the contraction factor α and propose an inexact triangular and skew-symmetric splitting (ITSS) iteration method for the numerical solutions of Markov chains. In Section 5, numerical experiments are presented to show the effectiveness of both the TSS and ITSS iteration methods. Finally, conclusions and future work are made in Section 6.

2. The regularized linear system

In this section, we will present a theorem to indicate that there exists a nonnegative constant $\epsilon \geq 0$ such that the matrix $A = \bar{A} + \epsilon I = Q^{\mathrm{T}} + \epsilon I$ is positive definite and consider a regularized linear system for solving the stationary probability vector of Markov chains.

Definition 1 (*See (1.2) p. 133 of [17]*)**.** Any matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ of the form

$$\mathcal{A} = sI - \mathcal{B}, \quad s > 0, \mathcal{B} \geq 0$$

is called an M-matrix when $s \geq \rho(\mathcal{B})$, the spectral radius of the nonnegative matrix $\mathcal{B}$. If $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is a nonsingular M-matrix, otherwise $\mathcal{A}$ is a singular M-matrix.

Definition 2 (*See (2.5) p. 141 of [17]*)**.** A symmetric nonsingular M-matrix is called a Stieltjes matrix.

As a matter of fact, Definition 2 implies that a symmetric nonsingular M-matrix $\mathcal{A}$ is a real symmetric positive-definite matrix with nonpositive off-diagonal entries. Here, we say that a nonsymmetric matrix $\mathcal{A}$ is positive definite if its symmetric part, i.e., $(\mathcal{A} + \mathcal{A}^{\mathrm{T}})/2$, is positive definite.

Theorem 3. *For any generator matrix $Q \in \mathbb{R}^{n\times n}$ as given in (1), there exists a nonnegative constant $\epsilon \geq 0$ such that the matrix $A = \bar{A} + \epsilon I = Q^{\mathrm{T}} + \epsilon I$ is positive definite.*

Proof. Our aim is to prove that the symmetric part of the matrix A, i.e., $(A + A^{\mathrm{T}})/2$, is positive definite. Let

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} \\ q_{31} & q_{32} & q_{33} & \cdots & q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} \end{pmatrix}, \quad q_{ii} > 0,\ q_{ij} \leq 0,\ 1 \leq i,\ j \leq n,\ i \neq j$$

and $a = \max(q_{ii}),\ 1 \leq i \leq n$, the largest number of its diagonal entries. Then, we have

$$\frac{Q + Q^{\mathrm{T}}}{2} = \begin{pmatrix} q_{11} & * & * & \cdots & * \\ * & q_{22} & * & \cdots & * \\ * & * & q_{33} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & q_{nn} \end{pmatrix} = aI - R,$$