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Qualitative and numerical analysis of the Rössler model: Bifurcations of equilibria

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ABSTRACT

In this paper, we show the combined use of analytical and numerical techniques in the study of bifurcations of equilibria of low-dimensional chaotic problems. We study in detail different aspects of the paradigmatic Rössler model. We provide analytical formulas for the stability of the equilibria as well as some of their codimension one, two, and three bifurcations. In particular, we carry out a complete study of the Andronov–Hopf bifurcation, establishing explicit formulas for its location and studying its character numerically, determining a curve of generalized-Hopf bifurcation, where the Hopf bifurcation changes from subcritical to supercritical. We also briefly study some routes among the different Andronov–Hopf bifurcation curves and how these routes are influenced by the local and global bifurcations of limit cycles. Finally, we show the U-shape of the homoclinic bifurcation curve at the studied parameter values.

1. Introduction

The Rössler model [1] is a paradigmatic problem among low-dimensional dynamical systems with chaotic behavior. So, a large number of articles [2–11] are still being published giving new partial results. However, this problem is not yet fully understood. The importance of this system, together with the Lorenz model, is that, being paradigmatic problems, they have become test problems for almost all new analytical and numerical techniques in computational dynamics.

The Rössler equations [1] are given by

$$\begin{aligned} \dot{x} &= -(y+z), \\ \dot{y} &= x+ay, \\ \dot{z} &= b+z(x-c), \end{aligned} \tag{1}$$

with $a, b, c \in \mathbb{R}$, and they are assumed to be positive and dimensionless.

The main goal of the present paper is to show how the use of numerical and analytical techniques can provide complete qualitative studies for low-dimensional chaotic problems, in particular in the study of bifurcations of equilibria. We focus our attention on the Rössler system in order to provide complete explicit expressions for the location and stability of the different equilibria, as well as for the Andronov–Hopf bifurcations. For the complete study of this bifurcation we need some numerical explorations due to the high complexity in the analysis of the codimension two bifurcations when the

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first Lyapunov coefficient vanishes. Also, we show how the global homoclinic bifurcations give rise to different routes in the connection via limit cycles between the different bifurcation curves in the space of parameters.

The paper is organized as follows. In Section 2, we provide analytical formulas for the stability of the equilibria. In Section 3, we present some codimension one, two, and three bifurcations of equilibria [12], in particular, a complete study of the Andronov–Hopf bifurcation is performed. In Section 4, we study different routes among the Hopf bifurcation curves and we show the connection among the bifurcation curves of equilibria, limit cycles, and the chaotic and regular regions, and we show the U-shape of the homoclinic bifurcation curve. Finally, in Section 5, we present some conclusions.

2. Equilibria: location and stability

Some of the first data to obtain in the analysis of a dynamical system are the equilibrium points and their bifurcations. The Rössler equations have two equilibrium points [7] for $c^2 > 4ab$, given by $P_1 = (-ap_1, p_1, -p_1)$ and $P_2 = (-ap_2, p_2, -p_2)$, with

$$p_1 := \frac{1}{2}\left(-\frac{c}{a} - \frac{\sqrt{c^2-4ab}}{a}\right), \qquad p_2 := \frac{1}{2}\left(-\frac{c}{a} + \frac{\sqrt{c^2-4ab}}{a}\right).$$

If $c^2 = 4ab$, then $P_1 = P_2$.

The stability of the equilibrium points can be studied analytically by means of the classical Routh–Hurwitz criterion, but in the literature there are only partial answers without explicit equations for the stable regions.

Proposition 1. *The equilibrium point P_1 in the Rössler system is always unstable and P_2 is linearly stable iff the parameters a, c belong to $S_1 = \{(a, c) | a \le 1$ and $c > 2a\}$ or to $S_2 = \{(a, c) | a \in (1, \sqrt{2})$ and $c \in (2a, 2a/(a^2 - 1))\}$, and the parameter b satisfies*

$$b_{\mathrm{H}}(a, c) \le b < b_{\mathrm{E}}(a, c), \tag{2}$$

with

$$\begin{aligned} b_{\mathrm{E}}(a,c) &:= \frac{c^2}{4a},\\ b_{\mathrm{H}}(a,c) &:= \frac{a\left(2 - a^4 + ca^3 + 2a^2 - ca + c^2 + (c-a)\sqrt{a^6 - 4a^4 + 2ca^3 - 4a^2 + c^2}\right)}{2\left(a^2+1\right)^2}. \end{aligned} \tag{3}$$

Proof. The proof is obtained by using the Routh–Hurwitz criterion (RHC). The RHC applied to a polynomial of degree 3, $x^3 + Ax^2 + Bx + C$, requires that $C > 0$, $A > 0$, and $AB > C$. Taking the linearized equations around both equilibrium points permits us to obtain their characteristic polynomials. For P_1 and P_2, respectively,

$$q_1(x) = x^3 + \frac{c - 2a - K}{2}x^2 + \frac{2a + c - a^2c + (1 + a^2)K}{2a}x - K,$$
$$q_2(x) = x^3 + \frac{c - 2a + K}{2}x^2 + \frac{2a + c - a^2c - (1 + a^2)K}{2a}x + K,$$

with $K = \sqrt{c^2 - 4ab}$. The point P_1 already fails the condition $C > 0$ for any value of the parameters.

For P_2, we have $C > 0$ when the equilibrium points exist. For $A > 0$, is necessary that $c \ge 2a$ or ($c \in (a, 2a)$ and $b < c-a$). The crucial point is the last condition: $AB > C$. After some algebra, we obtain the necessary and sufficient condition $-2a + 2b + 2a^2b + a^2c - ac^2 - a(c - a)K > 0$. For it to hold, first it is necessary that $-2a + 2b + 2a^2b + a^2c - ac^2 > 0$. Or, equivalently,

$$b > b_1(a, c) := \frac{a(2 + c(c - a))}{2(1 + a^2)}.$$

Also, it is required that

$$4(1 + a^2)^2b^2 - 4a(2 + 2a^2 - a^4 - ac + a^3c + c^2)b + 4a^2(1 + c(c - a)) > 0. \tag{4}$$

If $c < -a^3 + 2\sqrt{a^2 + a^4}$, then condition (4) is satisfied for all b. If $c = -a^3 + 2\sqrt{a^2 + a^4}$, then there is a number $b_H(a, c)$ such that condition (4) is true for all $b \ne b_H(a, c)$. Finally, if $c > -a^3 + 2\sqrt{a^2 + a^4}$, then there are two numbers $b_h(a, c) < b_H(a, c)$, and inequality (4) is satisfied for $b < b_h(a, c)$ or $b > b_H(a, c)$. But, as $b_h(a, c) < b_1(a, c)$, the only valid situation is $b > b_H(a, c)$.

If $c \in (a, 2a)$, then $b_1(a, c) < c - a$ implies that $c > a + 2/a > -a^3 + 2\sqrt{a^2 + a^4}$. So, $b_H(a, c)$ exists, and it is greater than $c - a$. Therefore, it is necessary that $c \ge 2a (> -a^3 + 2\sqrt{a^2 + a^4}$ for positive a).