Computers and Mathematics with Applications 62 (2011) 4161–4169

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Structure and reversibility of 2D hexagonal cellular automata ☆

Irfan Siap [a,*], Hasan Akin [b], Selman Uğuz [c]

[a] Department of Mathematics, Arts and Science Faculty, Yıldız Technical University, Istanbul, Turkey
[b] Department of Mathematics, Faculty of Education, Zirve University, 27260 Gaziantep, Turkey
[c] Department of Mathematics, Arts and Sciences Faculty, Harran University, 63120 Şanliurfa, Turkey

ARTICLE INFO

*Article history:*
Received 2 November 2010
Received in revised form 29 September 2011
Accepted 29 September 2011

*Keywords:*
Hexagonal cellular automata
Rule matrix
Reversible cellular automata
Matrix algebra

ABSTRACT

Cellular automata are used to model dynamical phenomena by focusing on their local behavior which depends on the neighboring cells in order to express their global behavior. The geometrical structure of the models suggests the algebraic structure of cellular automata. After modeling the dynamical phenomena, it is sometimes an important problem to be able to move backwards in order to understand it better. This is only possible if cellular automata is reversible. In this paper, 2D finite cellular automata defined by local rules based on hexagonal cell structure are studied. Rule matrix of the hexagonal finite cellular automaton is obtained. The rank of rule matrices representing the 2D hexagonal finite cellular automata via an algorithm is computed. It is a well known fact that determining the reversibility of a 2D cellular automata is a very difficult problem in general. Here, the reversibility problem of this family of 2D hexagonal cellular automata is also resolved completely.



## 1. Introduction and preliminaries

The hexagonal finite cellular automata (shortly HCA) are two dimensional (2D) cellular automata whose cells are of the form of a hexagonal. Morita et al. [1] introduced this type of cellular automaton (CA) and they called it hexagonal partitioned CA (HPCA). Hence, they studied logical universality of a reversible HPCA. They also stated that 2D HCAs are systems where identical finite automata are placed uniformly on the infinite hexagonal lattice space, such that they synchronously change their states by communicating with neighboring cells. The hexagonal cell arrangement of 2D automata has been observed mainly on natural dynamical events. In the following, we mention some but surely not all of the latest applications of 2D hexagonal cellular automata. A remarkable application of the family of these CAs is presented by Trunfio [2], where a model is presented to simulate the evolution of forest fires and Hernández Encinas et al. [3], where they introduce a new mathematical model for predicting the spread of a fire front in homogeneous and inhomogeneous environments. They have shown that cells stand for regular hexagonal areas of the forest. Further, they have concluded that all graphical models obtained in their study are seen to be in agreement with the experience of fire spreading in real forests. Also, in [4], debris flows are simulated and modeled by 2D hexagonal cellular automata. These family of cellular automata are also applied to design discrete models of chemical reaction–diffusion systems [5]. In [6], a graphics processor unit (GPU)-accelerated method for real-time computing and rendering cellular automata (CA) applied to hexagonal grids is proposed and the advantage of this is discussed. The authors in [6] present a model for hexagonal grid algorithms which encodes and transmits large CA key-codes

☆ This paper is partially presented at ICMS-International Conference on Mathematical Sciences, 23–27 Nov. 2010, Bolu, Turkey, Akın et al. (2010) [27]. This work is supported by TUBITAK, Project Number: 110T713.

* Corresponding author.
*E-mail addresses:* isiap@yildiz.edu.tr (I. Siap), hasanakin69@gmail.com (H. Akin), selmanuguz@gmail.com (S. Uğuz).

0898-1221/$ – see front matter 
doi:10.1016/j.camwa.2011.09.066

to the graphics card. Also, 2D CA have found applications in traffic modeling. For instance, multi-value (including ternary) CA models for traffic flow are proposed in [7]. Recently, cellular automata have found applications in cryptography [8,9], especially 2D CA have been proposed for a multisecret sharing scheme for colored images [10].

Recently, due to these applications and modelings, the algebraic structure of cellular automata has been of much interest to the researchers [11,12]. In [13], an extension of the well known Hedlund's theorem is studied by generalizing the notion of local rule in the definition of cellular automata. Further, the behavior of the linear 2D CA over binary fields ($Z_2 = \{0, 1\}$) by using matrix algebra setting has been studied [14–17]. Further, Das [18] has studied the characterization of 1D CA by means of matrix algebra. Khan et al. [15] developed an analytical tool to study all the nearest neighborhood 2D CA linear transformations. They proposed a new rule convention to divide the 2D linear CA and tried to study the characterization of that 2D CA with different rules. Also, in [19], characterizations of 2D hybrid cellular automata over binary fields is presented. Further in [20], 2D cellular automata are applied to text compression.

In [21], the authors have characterized a 2D finite CA by using matrix algebra built on $Z_3 = \{0, 1, 2\}$ and have analyzed some results about the rule numbers 2460N and 2460P. In [22], the authors have obtained necessary and sufficient conditions for the existence of Garden of Eden configurations for 2D ternary CA and have provided an algorithm to obtain the number of Garden of Eden configurations for the 2D CA defined by rule 2460N.

Algebraic representation of 2D CA helps in determining the characterization of CA. An important characterization is the determination of the reversibility of CA. Kari [23] has proved that the reversibility of a CA with dimension larger or equal to two is not decidable. In other words, due to its complexity, Kari has shown that the inverse of a given CA with higher dimension cannot be found by an algorithm in general. Further, Durand in [24] shows that the problem of finding the inverse of a 2D CA is a very difficult problem. Recently, the reversibility problem for elementary cellular automata with rule number 150 has been studied [25].

In this paper, we deal with CA defined by a hexagonal local rule over the ternary field $Z_3$. We obtain the rule matrix of the hexagonal finite cellular automaton (HFCA). We compute the rank of rule matrices related to HFCA via an algorithm. Hence, we determine the reversibility of this type 2D CA which is one of the difficult problems in higher dimension as explained in the previous paragraph. Further, by using the matrix algebra it is shown that the HFCA is reversible if the number of columns is even and not reversible if the number of the columns is odd.

Now, we present 2D HFCA and their rule numbering. Since the tiling of a plane is done by hexagonal cells, we need to make clear the reference address (or labeling) of the cells. So we distinguish them by separating it into two cases as follows: depending on the neighborhood structure of the extreme cells, mainly there exist two approaches:

- A Periodic Boundary CA is the one where the extreme cells in the boundaries are adjacent to each other periodically.
- A Null Boundary CA is the one where the extreme cells in the boundaries are connected to the zero states. The surrounding cells are all in zero state. For instance, see Fig. 2.

In this paper, we deal with CA defined by hexagonal rules under Null Boundary Condition (NBC). For convenience of analysis, the state of each cell is an element of a finite or an infinite state set. Moreover, the state of the cell $(i, j)$ at time $t$ is denoted by $x^{(t)}_{(i,j)}$. The state of the cell $(i, j)$ at time $t + 1$ is denoted by $x^{(t+1)}_{(i,j)} = y^{(t)}_{(i,j)}$. Let us consider the hexagonal information matrix $C^{(t)} = \begin{pmatrix} x^{(t)}_{11} & \dots & x^{(t)}_{1n} \\ \vdots & \ddots & \vdots \\ x^{(t)}_{m1} & \dots & x^{(t)}_{mn} \end{pmatrix}$. The matrix $C^{(t)}$ is called the configuration of the 2D finite CA at time $t$. We associate planar hexagonal presentations with column vectors by transforming them from $C^{(t)}$ to $([X]_{mn\times 1})^T = (x^{(t)}_{11}, x^{(t)}_{12}, \dots, x^{(t)}_{1n}, \dots, x^{(t)}_{m1}, \dots, x^{(t)}_{mn})^T$. Hence, we can consider the transition matrix $T_R$ such that $(T_R)_{mn\times mn}.[X]_{mn\times 1} = [Y]_{mn\times 1}$, where $([Y]_{mn\times 1})^T = (y^{(t)}_{11}, y^{(t)}_{12}, \dots, y^{(t)}_{1n}, \dots, y^{(t)}_{m1}, \dots, y^{(t)}_{mn})^T$.

If $j$ is an even positive integer (Fig. 1-(a)), then we have

$$y^{(t)}_{(i,j)} = ax^{(t)}_{(i-1,j)} + bx^{(t)}_{(i,j+1)} + cx^{(t)}_{(i+1,j+1)} + dx^{(t)}_{(i+1,j)} + ex^{(t)}_{(i+1,j-1)} + fx^{(t)}_{(i,j-1)} \mod 3. \tag{1}$$

If $j$ is an odd positive integer (Fig. 1-(b)), then we have

$$y^{(t)}_{(i,j)} = ax^{(t)}_{(i-1,j)} + bx^{(t)}_{(i-1,j+1)} + cx^{(t)}_{(i,j+1)} + dx^{(t)}_{(i+1,j)} + ex^{(t)}_{(i,j-1)} + fx^{(t)}_{(i-1,j-1)} \mod 3, \tag{2}$$

where $a, b, c, d, e, f \in Z_3^* = Z_3 \setminus \{0\} = \{1, 2\}$ and $x^{(t)}_{(i,j)} \in Z_3$.

## 2. The rule matrix of 2D HFCA with hexagonal rule

In this section, we obtain the rule matrix of 2D finite CA with hexagonal rule over the field $Z_3$ under NBC (see [3] for the details about the rule matrix). As we have set the labeling of the cells above, now we need to relate the states of these cells. In Fig. 2, a sample state of order $3 \times 4$ is given. At a particular state $t$, we consider a hexagonal information matrix representing the state of order $m \times n$ with entries from $Z_3$, say $C^{(t)}$. The next $t + 1$ is also again a hexagonal information matrix of order $t + 1$, say $C^{(t+1)}$. Due to the geometry and the relations of the neighborhoods, a change from the $t$th state to $t + 1$th state is a function from matrix space of order $m \times n$ with coefficients in $Z_3$, denoted by $M_{n\times m}(Z_3)$ in its self. It is well