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Regularity for nonlinear variational inequalities of hyperbolic type[☆]

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ABSTRACT

In this paper, we deal with the regularity for nonlinear variational inequalities of second order in Hilbert spaces with more general conditions on the nonlinear terms and without condition of the compactness of the principal operators. We also obtain the norm estimate of a solution of the given nonlinear equation on $C([0,T];V)\cap C^1((0,T];H)\cap C^2((0,T];V^*)$ by using the results of its corresponding hyperbolic semilinear part.

1. Introduction

The subject of this paper is to consider the initial value problem of the following nonlinear variational inequalities of second order in Hilbert spaces;

$$\begin{cases} (u''(t)+Au(t),u(t)-z)+\phi(u(t))-\phi(z) \\ \quad \le (f(t,u(t))+k(t),u(t)-z), \text{ a.e.}, \quad \forall z\in V \\ u(0)=u^0, \qquad u'(0)=u^1. \end{cases} \tag{NVE}$$

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gaarding's inequality, and let $\phi\colon V\to(-\infty,+\infty]$ be a lower semicontinuous, proper convex function. The nonlinear term $f(\cdot,u)$, which is a locally Lipschitz continuous operator with respect to u from V to H, is a semilinear version of the quasilinear one considered in [1–3], and a forcing term $k\in L^2(0,T;V^*)$. By the definition of the subdifferential operator $\partial\phi$, the problem (NVE) is represented by the following nonlinear functional differential problem:

$$\begin{cases} u''(t)+Au(t)+\partial\phi(u(t))\ni f(t,u(t))+k(t), \quad 0<t, \\ u(0)=u^0, \qquad u(s)=u^1. \end{cases} \tag{NDE}$$

The background of these variational problems is found in physics, especially in solid mechanics, where nonconvex and multi-valued constitutive laws lead to differential inclusions. We refer to [4–8] to see the applications of differential inclusions. There are extensive literatures on parabolic variational inequalities of first order and the Stefan problems (see [9,10] and the book by Duvaut and Lions [11]). However there are few papers treating the variational inequalities of second order with nonlinear inhomogeneous terms.

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In this paper we are primarily interested in the regular problem for the variational inequalities of second order with nonlinear inhomogeneous terms that arise as direct consequences of the general theory developed previously, and we consider to put in perspective those models of initial value problems which can be formulated as nonlinear differential equations of variational inequalities. The approach used here is similar to that developed in Yosida [12] in which more general hyperbolic equations are also treated. When the nonlinear mapping k is locally Lipschitz continuous from $\mathbb{R} \times V$ into H, we will obtain that the majority of the regularity for parabolic variational inequalities of first order can also applicable to (NDE) with nonlinear perturbations (see [9–11,13,14,12]).

Section 2 gives some basic properties on the principal operator A to consider a representation formula of solutions for the general hyperbolic semilinear equations in case $\phi \equiv 0$ [9–11,13,15,12]. In Section 3, we will introduce single valued smoothing systems corresponding to nonlinear variational inequalities (NDE) by using approximate function $\phi_\epsilon(x) = \inf\{\|x-y\|_*^2/2\epsilon + \phi(y): y \in H\}$ (see [9,10]).

Section 4 deals with the wellposedness for solutions of (NDE) by converting the problem into the contraction mapping principle with more general conditions on the nonlinear terms and without conditions of the compactness of the principal operators, and obtain the norm estimate of a solution of the above nonlinear equation on $C([0,T];V) \cap C^1((0,T];H) \cap C^2((0,T];V^*)$ by using the results of its corresponding the hyperbolic semilinear part in case $\phi \equiv 0$ as seen in [13].

2. Parabolic variational inequalities

Let H be a complex Hilbert space with inner product $(,)$ and norm $|\cdot|$. Let V be embedded in H as a dense subspace with inner product and norm by $((,))$ and $\|\cdot\|$, respectively. By considering $H = H^*$. We may write $V \subset H \subset V^*$ where H^* and V^* denote the dual spaces of H and V, respectively. For $l \in V^*$ we denoted (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$\|l\|_* = \sup_{v \in V} \frac{|(l,v)|}{\|v\|}.$$

Therefore, we assume that V has a stronger topology than H and, for the brevity, we may regard that

$$\|u\|_* \le |u| \le \|u\|, \quad \forall u \in V.$$

Definition 2.1. Let X and Y be complex Banach spaces. An operator S from X to Y is called antilinear if $S(u+v) = S(u)+S(v)$ and $S(\lambda u) = \bar{\lambda} S(u)$ for $u, v \in X$ and for $\lambda \in \mathbb{C}$.

Let $a(u,v)$ be a quadratic form defined on $V \times V$ which is linear in u and antilinear in v.

We make the following assumptions: (i) $a(u,v)$ is bounded, i.e. $\exists c_0 > 0$ such that

$$|a(u,v)| \le c_0 \|u\| \cdot \|v\|; \tag{2.1}$$

(ii) $a(u,v)$ is symmetric, i.e.

$$a(u,v) = \overline{a(v,u)};$$

(iiii) $a(u,v)$ satisfies the Gaarding's inequality, i.e.

$$\text{Re}\, a(u,u) \ge \delta \|u\|^2, \quad \delta > 0. \tag{2.2}$$

Let A be the operator such that $(Au, v) = a(u,v)$ for any $u, v \in V$. Then, as seen in Theorem 2.2.3 of [16], the operator A is positive definite and self-adjoint, $D(A^{1/2}) = V$, and

$$a(u,v) = (A^{1/2}u, A^{1/2}v), \quad u, v \in V.$$

It is also known that the operator A is a bounded linear from V to V^*. The realization of A in H which is the restriction of A to $D(A) = \{v \in V : Av \in H\}$ is denoted by A_H, which is structured as a Hilbert space with the norm $\|v\|_{D(A)} = |A_H v|$. Then the operators A_H and A generate analytic semigroups in both of H and V^*, respectively. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one which continuous injection.

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers whose norms are integrable and $W^{m,p}(0,T;X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0,T;X)$. $C^m([0,T];X)$ is the set of all m-times continuously differentiable functions from $[0,T]$ into X. Let X and Y be complex Banach spaces. Denote by $B(X,Y)$ (resp. $\overline{B}(X,Y)$) the set of all bounded linear (resp. antilinear) operators from X to Y. Let $B(X) = B(X,X)$.

We consider the initial value problem of the following variational inequality

$$\begin{cases} (u''(t) + Au(t), u(t) - z) + \phi(u(t)) - \phi(z) \\ \quad \le (f(t,u(t)) + k(t), u(t) - z), \text{ a.e.}, \quad \forall z \in V \\ u(0) = u^0, \qquad u(s) = u^1. \end{cases} \tag{NVE}$$