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Periodic BVPs in ODEs with time singularities

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ABSTRACT

In this paper, we show the existence of solutions to a nonlinear singular second order ordinary differential equation,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)),$$

subject to periodic boundary conditions, where $a > 0$ is a given constant, $\lambda > 0$ is a parameter, and the nonlinearity $f(t, x, y)$ satisfies the local Carathéodory conditions on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Here, we study the case that a well-ordered pair of lower and upper functions does not exist and therefore the underlying problem cannot be treated by well-known standard techniques. Instead, we assume the existence of constant lower and upper functions having opposite order. Analytical results are illustrated by means of numerical experiments.

1. Introduction

Singular periodic boundary value problems for the ordinary differential equation $u''(t) = g(t, u(t), u'(t))$, where $g(t, x, y)$ shows singularities in the phase variable x, have been widely studied for more than 40 years and there exists rich literature on their properties.

In this paper, we focus our attention on another type of singular periodic problems, namely on those with singularities in the time variable t. In many applications, cf. [1–4], second order singular models are posed on the interval $(0, T)$ and have the form,

$$u''(t) = \frac{a_1}{t^\alpha}u'(t) + \frac{a_0}{t^{\alpha+1}}u(t) + f(t, u(t), u'(t)), \qquad u''(t) = \frac{a}{t^\alpha}u'(t) + f(t, u(t), u'(t)),$$

where a_1, a_0, a and f are given. We say that for $\alpha = 1$, the problem exhibits a time singularity of the first kind at $t = 0$, while for $\alpha > 1$, the time singularity is essential or of the second kind.

Let $T > 0$, and let us, by $L_1[0, T]$, denote the set of functions which are (Lebesgue) integrable on $[0, T]$. The corresponding norm is given by $\|u\|_1 := \int_0^T |u(t)|dt$. Moreover, let us, by $C[0, T]$ and $C^1[0, T]$, denote the sets of functions being continuous on $[0, T]$, and having continuous first derivatives on $[0, T]$, respectively. The maximum norm on $C[0, T]$ is defined as $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$. We denote by $AC[0, T]$ and $AC^1[0, T]$ the sets of functions which are absolutely continuous on $[0, T]$, and which have absolutely continuous first derivatives on $[0, T]$, respectively. Analogously, $AC^1_{loc}(0, T]$ is the set of functions having absolutely continuous first derivatives on each compact subinterval of $(0, T]$.

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In this paper, we investigate the following parameter dependent ordinary differential equation, with a singularity of the first kind,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)), \tag{1}$$

where $a > 0$, $\lambda > 0$, and the function $f(t, x, y)$ is defined for a.e. $t \in [0, T] \subset \mathbb{R}$ and for all $(x, y) \in \mathcal{D} \subset \mathbb{R} \times \mathbb{R}$. Clearly, the above equation is singular at $t = 0$ because of the first term in the right-hand side, which is in general unbounded for $t \to 0$. Moreover, we also allow the function f to be unbounded or bounded but discontinuous for certain values of the time variable $t \in [0, T]$. This form of f is motivated by a variety of initial and boundary value problems known from applications and having nonlinear, discontinuous forcing terms, such as electronic devices which are often driven by square waves or more complicated discontinuous inputs. Typically, such problems are modeled by differential equations where f has jump discontinuities at a discrete set of points in $(0, T)$, cf. [5]. Many other applications (cf. [1–3,6–15]) show similar structural difficulties. This motivates to assume the following properties of f in (1):

(A_1): $f(\cdot, x, y) : [0, T] \to \mathbb{R}$ is measurable for all $(x, y) \in \mathbb{R}^2$ and $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [0, T]$.
(A_2): For a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R}^2$ the estimate

$$|f(t, x, y)| \le g(t)\omega(|y|)$$

holds with positive functions $g \in L_1[0, T]$ and $\omega(y) \in C[0, \infty)$, where ω is nondecreasing.

Definition 1. A function $u : [0, T] \to \mathbb{R}$ is called a solution of Eq. (1) if $u \in AC^1[0, T]$ and

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t))$$

holds a.e. on $[0, T]$.

In the sequel, we study the differential equation (1) subject to periodic boundary conditions,

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{2}$$

In the literature, there are only very few results for periodic boundary value problems of the form

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)), \qquad u(0) = u(T),\ u'(0) = u'(T), \tag{3}$$

with a non-integrable singularity in the time variable (see [16,17,12,13]) where existence results for this type of problems are shown using the lower and upper functions technique. All these results are obtained under the assumption that there exists a pair of constant well-ordered lower and upper functions. In this paper, we show the existence result for problem (3) in the dual and more difficult case, where the problem has *the opposite-ordered upper and lower functions*, which means that the upper function lies below the lower one. We illustrate this situation with examples to whom the earlier known results do not apply.

The paper is organized as follows. In order to prove that the periodic problem (3) is solvable, we first show in Section 3 that the auxiliary problem,

$$u''(t) = \frac{a}{t}u'(t) + \lambda f(t, u(t), u'(t)), \quad u(0) = u(T),\ u'(T) = 0, \tag{4}$$

has a solution. The main tool in the proof is the Leray–Schauder degree method; cf. e.g., [18]. Then, applying Lemma 1 from Section 2, we conclude that for solutions u of (4) also $u'(0) = 0$ holds. Hence, the solution u of (4) satisfies (2) and consequently, is a solution of problem (3). We give two examples of boundary value problem (3), which can be analyzed using results developed in this paper and for which on the other hand, a pair of constant well-ordered lower and upper functions does not exist. Finally, in Section 4, we illustrate the theoretical findings by means of numerical experiments.

2. Preliminaries

The following two results are used in the next section in the discussion of the boundary value problem (3).

Lemma 1. *Let* (A_1) *and* (A_2) *hold and let* $\lambda > 0$, $a \neq 0$. *Suppose that* $u \in AC^1_{loc}(0, T]$ *satisfies Eq.* (1) *a.e. on* $[0, T]$ *and*

$$\sup\{|u(t)| + |u'(t)| : t \in (0, T]\} < \infty.$$

Then $\lim_{t\to 0_+} u'(t) = 0$.

Proof. The proof follows from [14, Corollary 3.5]. □