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Two-dimensional finite deformations evaluated from pre- and post-deformation markers: Application to balanced cross sections

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ABSTRACT

A method is proposed for determining the 2D deformation gradient tensor that represents the deformations of pre- and post-deformation markers with arbitrary shapes. The deformation is not necessarily coaxial. The tensor is evaluated in a least-square sense. Therefore, the method can deal with heterogeneous deformations, and calculate their average tensor. The inverse method has the residuals that can be directly converted to the logarithmic strain needed to transform the calculated post-deformation shapes to observed ones. In addition, we propose the measure of heterogeneity for finite deformations. The method was applied to artificial and natural data from balanced cross sections.

1. Introduction

Observation of tectonic deformations allows us to test the numerical or sandbox models that predicts deformation field (e.g., McKenzie, 1979; Erslev, 1991; Allmendinger, 1998; Iwamori, 2003; Poblet and Bulnes, 2007; Henk and Nemčok, 2008; Graveleau et al., 2012). Zhang and Hynes (1995) proposed a method to determine non-coaxial 3D deformation within such a shear zone that did not have shearing in the direction across the zone.

Here, we propose a theoretical method to determine two-dimensional, deformation gradient tensor, $\boldsymbol{F}$, which represents not only coaxial but also non-coaxial deformations. The tensor can always be decomposed into left-stretch tensor and orthogonal tensor, $\boldsymbol{F} = \boldsymbol{VR}$ (Spencer, 2004). Strain ellipse or ellipsoid depicts only the left-stretch tensor, $\boldsymbol{V}$. Knowledge of strain ellipse or ellipsoid is not enough to determine $\boldsymbol{F}$, because $\boldsymbol{R}$ is remain undetermined (Davis and Titus, 2011, p. 1052). Popular strain analysis methods such as R_f/ϕ techniques are insufficient. Accordingly, Zhang and Hynes (1995) utilized the orientation of a shear plane and the direction of shear to solve for $\boldsymbol{F}$. We compare pre- and post-deformation shapes of markers to determine $\boldsymbol{F}$ without assuming directions of tectonic motions.

In this work, the ellipse fitting technique of Teagure (1980) was used for this purpose: Mulchrone and Choudhury (2004) proved that the technique is useful for geological strain analysis. Using the technique, we can deal with the coaxial and non-coaxial deformations of markers with arbitrary shapes. It is assumed that the markers were subject to the same deformation at least approximately. Strictly, deformation field is heterogeneous, but a coarse graining approach (e.g., Lesne, 2006) allows rough but quantitative estimation of tectonic deformations (Fig. 1). We propose a measure of heterogeneity to evaluate how this assumption is valid for a given data set. Assuming steady incremental deformation, Ramberg (1975) drew deformation trajectories by the numerical integration of velocity gradient tensor. We introduce a method to draw them without the integration.

Pre-deformation shapes are rarely known in nature. However, they can be inferred in balanced cross-sections (e.g., Woodward et al., 1989). The sections place quantitative constraints on the long-term tectonic deformations. We applied our method to such sections. We ignore area changes during deformation. The validity of this treatment is discussed in the final section. The areas of deformation markers in a section are used only as the weights of data in the mathematical inversion to determine the deformation gradient tensor that represent the shape changes of the markers. The numerical examples for the present method are presented using artificial and natural data sets in §5. The important symbols used in this paper is listed in Table 1.

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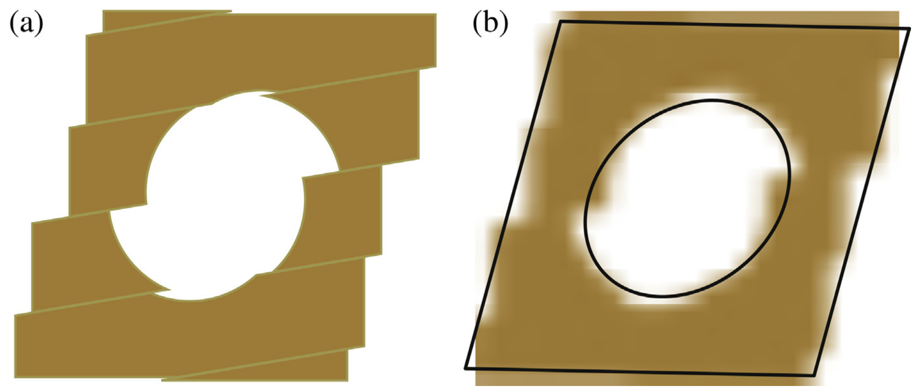

Fig. 1. (a) Schematic configuration of heterogeneous deformation. (b) The deformation is approximated to a homogeneous one (solid lines) by coarse graining.

2. Formulation

2.1. Problem statement

The deformations that affect parts of a rock body are assumed to be represented by a deformation gradient tensor, $\boldsymbol{F}$. This tensor is defined as the transformation matrix of the pre- and post-deformation position vectors,

$$\boldsymbol{X} = \boldsymbol{F}\boldsymbol{\Xi}, \tag{1}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{X}$ are the vectors, respectively. $\boldsymbol{F}$ is also called position gradient tensor (e.g., Zhang and Hynes, 1995), meaning that not the absolute but relative positions are important for describing deformation. That is, the origin of the position vectors can be chosen arbitrarily.

To determine $\boldsymbol{F}$, we use the ellipses fitted to the pre- and post-deformation configurations through Teague's (1980) technique. The problem we are tackling is how $\boldsymbol{F}$ is determined using the n pairs of pre- and post-deformation ellipses, each of which is characterized by its aspect ratio, R, and major-axis orientation, ϕ, with respect to a reference orientation in the plane where deformation is considered. We neglect area changes during deformation. This condition is written as det $\boldsymbol{F} = 1$, because this determinant denotes volume change (Spencer, 2004, p. 93).

Suppose that we have the n pairs of ellipses that represent pre- and post-deformation markers, and let $(R_\mathrm{i}^{(k)}, \phi_\mathrm{i}^{(k)})$ and $(R_\mathrm{f}^{(k)}, \phi_\mathrm{f}^{(k)})$ be the paired data of the kth ellipse, where the subscripts 'i' and 'f' distinguish the pre- and post-deformation quantities. The problem we consider, here, is how $\boldsymbol{F}$ is determined from $(R_\mathrm{i}^{(1)}, \phi_\mathrm{i}^{(1)}), \ldots (R_\mathrm{i}^{(n)}, \phi_\mathrm{i}^{(n)})$ and $(R_\mathrm{f}^{(1)}, \phi_\mathrm{f}^{(1)}), \ldots (R_\mathrm{f}^{(n)}, \phi_\mathrm{f}^{(n)})$ with the constraint, det $\boldsymbol{F} = 1$.

Table 1
List of symbols. The upright roman subscripts, 'i' and 'f,' are used to distinguish the quantities of pre-deformation (initial) and post-deformation (final) states, respectively.

$A^{(k)}$	Area of the kth deformation marker	
$d_\mathrm{H}()$	Hyperbolic distance	Eq. (5)
Γ	Angle of elevation of $\boldsymbol{p}$	Fig. 4
$\boldsymbol{F}$	Deformation gradient tensor	Eq. (1)
$\boldsymbol{F}_\mathrm{p}$	Deformation gradient tensor corresponding to pure shear	Eq. (12)
$\boldsymbol{F}_\mathrm{r}$	Deformation gradient tensor corresponding to rigid-body rotation	Eq. (10)
H	Heterogeneity	Eq. (17)
H^2	Unit hyperboloid	Eq. (2)
$\boldsymbol{I}$	Identity tensor	
$\boldsymbol{J}$	Minkowski tensor	Eq. (7)
$\mathbf{M}^3$	Three-dimensional Minkowski space	
n	Number of deformation markers	
$\boldsymbol{p}$	Pole vector of a plane in M^3	Fig. 4
$\boldsymbol{R}$	Aspect ratio of an ellipse	
$\boldsymbol{T}_\mathrm{p}$	Transformation matrix in M^3 corresponding to pure shear in the physical space	Eq. (11)
$\boldsymbol{T}_\mathrm{r}$	Transformation matrix in M^3 corresponding to rigid-body rotation in the physical space	Eq. (9)
x_0, x_1, x_2	Rectangular Cartesian coordinates in M3	Fig. 2
ϕ	Major-axis orientation of an ellipse	
ρ	Radial coordinate on H^2	Eq. (3), Fig. 2
ψ	Tangential coordinate in M^2	Eq. (3), Fig. 2
$\circ$	Lorentzian inner product	Eq. (6)
$*$	Lorentzian outer product	Eq. (13)
$\Vert\ \Vert$	Minkowski norm	Eq. (8)

2.2. Necessary conditions for the parameter space of inversion

Given the tensor, $\boldsymbol{F}$, it is possible to calculate the aspect ratio and major-axis orientation $(R_\mathrm{c}^{(k)}, \phi_\mathrm{c}^{(k)})$ of the ellipse that is derived from $(R_\mathrm{i}^{(k)}, \phi_\mathrm{i}^{(k)})$ (Section 3.2). Accordingly, $\boldsymbol{F}$ can be determined by mathematical inversion. That is, the optimal $\boldsymbol{F}$ is determined by minimizing the sum of the dissimilarities or distances between $(R_\mathrm{c}^{(k)}, \phi_\mathrm{c}^{(k)})$ and $(R_\mathrm{f}^{(k)}, \phi_\mathrm{f}^{(k)})$. The distance indicates the residual of the optimal solution. Then, how do we define the distances?

Objective methods for estimation of model parameters require optimization of a cost function, representing a measure of distance between the observations and the corresponding model predictions (e.g., Ebtehaj et al., 2010). The naive answer to the question is to use the quantity, $d = [(R_\mathrm{f} - R_\mathrm{c})^2 + (\phi_\mathrm{f} - \phi_\mathrm{c})^2]^{1/2}$, as the distance for the kth deformation marker. This answer implicitly use the Euclidean plane with the rectangular Cartesian coordinates, R and ϕ, with d being Euclidean distance. The quantity is a bad distance measure (Yamaji, 2008, 2013), because any difference in ϕ does not make sense for the case of $R = 1$. The significance of this difference becomes greater with increasing R. In addition, the ellipses with the same aspect ratio but different ϕ values, 0 and 180°, are identical, though the ϕ values are different.

If two ellipses are represented by the symbols, E^1 and E^2, their distance, $d\,(E^1, E^2)$, must satisfy the five conditions (Yamaji and Sato, 2006):

1. Non-negativity: $d(E^1, E^2) \geq 0$.
2. Identity of indiscernibles: $d\,(E^1, E^2) = 0$ if and only if $E^1 = E^2$.
3. Symmetry: $d\,(E^1, E^2) = d\,(E^2, E^1)$.
4. Triangle inequality: $d(E^1, E^3) \leq d(E^1, E^2) + d(E^2, E^3)$, where E^3 represent an ellipse.
5. Invariance: $d\,(E^1, E^2)$ is independent from the choice of reference orientation.