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Precise asymptotics in the law of logarithm under dependence assumptions[☆]

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ABSTRACT

In a recent paper by Spătaru [Precise asymptotics for a series of T.L. Lai, Proc. Amer. Math. Soc. 132 (11) (2004) 3387–3395] a precise asymptotics in the law of the logarithm for sequence of i.i.d. random variables has been established. In this paper we show that there is an analogous result for strictly stationary φ-mixing sequence. To prove this result, we have to use a different method. One of our main tools is the Gaussian approximation technique.

1. Introduction and results

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution F. Let $S_n = \sum_{k=1}^{n} X_k$ and suppose that $\mathsf{E}X = 0$ and $0 < \mathsf{E}X^2 = \sigma^2 < \infty$. There is a lot of literature concerning precise asymptotic behavior of the partial sum S_n. The first such result is due to Heyde [1], who proved that $\lim_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geq n\varepsilon) = \sigma^2$. This result was extended in [2], wherein the author proved a more general theorem. In a recent paper, assuming the distribution of X is attracted to a stable distribution with exponent $\alpha > 1$, Spătaru [3] proved $\sum_{n=1}^{\infty} \frac{1}{n}\mathsf{P}(|S_n| \geq n\varepsilon) \sim \frac{\alpha}{\alpha-1}(-\log\varepsilon)$ as $\varepsilon \searrow 0$. After this interesting contribution, more and more authors have devoted their efforts to the work of precise asymptotics; see [4–8], for example.

There already exist some classical methods to deal with the precise asymptotics for the case of "$\varepsilon \searrow 0$". Such results are usually not too difficult to derive. However, for the case of "$\varepsilon \searrow c_0$" with some positive number c_0, powerful tools and finer arguments are needed. For example, by using a non-uniform estimate in the normal approximation, Li, Wang and Rao [9] obtained much more general results on precise asymptotics in the law of the iterated logarithm for the i.i.d. case. Using the Berry–Esseen inequality, Spătaru [7] obtained the precise asymptotics in the law of logarithm. His result is as follows.

Theorem A. *Let $\{X, X_1, X_2, \ldots\}$ be a sequence of i.i.d random variables, $1 < r < 3/2$, $\mathsf{E}X^2 = \sigma^2$ and $\mathsf{E}(X^{2r}/(\log^+|X|)^r) < \infty$. Then*

$$\lim_{\varepsilon\searrow\sigma\sqrt{2(r-1)}} \sqrt{\varepsilon^2 - 2\sigma^2(r-1)} \sum_{n=2}^{\infty} n^{r-2}\mathsf{P}\left(|S_n| \geq \varepsilon\sqrt{n\log n} + a_n\right) = \sigma\sqrt{\frac{2}{r-1}},$$

where $a_n = o(\sqrt{n/\log n})$.

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(Theorem A is exactly the same as Theorem 1' of [7], in which a_n should be $a_n = o(\sqrt{n/\log n})$; see the proof of [7].)

It is well-known that the rate in central limit theorem is of the order of $O(n^{-1/2})$ and can not be improved. Therefore, by examining the proof in [7], one can find that it is difficult to extend Theorem A to the case of $r \geq 3/2$ by his method. Moreover, the rate of the central limit theorem for mixing random variables is not as sharp as that of the i.i.d. case. Hence we shall develop a different method to extend Theorem A to the case of $r \geq 3/2$. Meanwhile an analogous result to Theorem A will also be derived for mixing random variables. Our method is based on a coupling lemma (see Lemma 2.1) and the Gaussian approximation technique due to Sakhaneko [10]. The latter approximation was used by Zhang [11] for obtaining the sufficient and necessary conditions of the precise rates in the law of iterated logarithm for the i.i.d. random variables.

Now we give some definitions of mixing random variables. Let $\Im_a^b$ denote the σ-field generated by $X_a, X_{a+1}, \ldots, X_b$ and define

$$\varphi(\Im_1^k, \Im_{k+n}^\infty) := \sup\{|\mathsf{P}(B|A) - \mathsf{P}(B)|; A \in \Im_1^k, B \in \Im_{k+n}^\infty\},$$
$$\varrho(\Im_1^k, \Im_{k+n}^\infty) := \sup\{|\mathrm{corr}(U, V)|; U \in L^2(\Im_1^k), V \in L^2(\Im_{k+n}^\infty)\},$$
$$\varphi(n) := \sup_{k\geq 1} \varphi(\Im_1^k, \Im_{k+n}^\infty), \qquad \varrho(n) := \sup_{k\geq 1} \rho(\Im_1^k, \Im_{k+n}^\infty).$$

A sequence $\{X_j\}_{j\geq 1}$ of random variables is called φ-mixing if $\varphi(n) \to 0$ and ρ-mixing if $\varrho(n) \to 0$. It is known that $\varrho(n) \leq 2\varphi^{1/2}(n)$ and hence a φ-mixing sequence is ρ-mixing.

We state our results as follows.

Theorem 1.1. *Let $1 < r < 3/2$. Let $\{X, X_n; n \geq 1\}$ be a strictly stationary φ-mixing sequence such that*

$$\varphi(n) = O\left(\frac{1}{n^T}\right) \quad \textit{for some } T > 2,$$

and

$$\mathsf{E}X = 0, \qquad \mathsf{E}(X^{2r}/(\log^+ |X|)^r) < \infty.$$

Then

$$\lim_{\varepsilon \searrow \sqrt{2(r-1)}} \sqrt{\varepsilon^2 - 2(r-1)} \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\left(|S_n| \geq \varepsilon\sqrt{\mathsf{E}S_n^2 \log n} + a_n\right) = \sqrt{\frac{2}{r-1}},$$

whenever $a_n = O(\sqrt{n}/(\log n)^\gamma)$ for some $\gamma > 1/2$.

Theorem 1.2. *Let $r \geq 3/2$. Let $\{X, X_n; n \geq 1\}$ be a strictly stationary φ-mixing sequence such that*

$$\varphi(n) = O\left(\frac{1}{n^T}\right) \quad \textit{for some } T > 2r - 1,$$

and

$$\mathsf{E}X = 0, \qquad \mathsf{E}(X^{2r}(\log^+ |X|)^r) < \infty.$$

Then

$$\lim_{\varepsilon \searrow \sqrt{2(r-1)}} \sqrt{\varepsilon^2 - 2(r-1)} \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\left(|S_n| \geq \varepsilon\sqrt{\mathsf{E}S_n^2 \log n} + a_n\right) = \sqrt{\frac{2}{r-1}},$$

whenever $a_n = O(\sqrt{n}/(\log n)^\gamma)$ for some $\gamma > 1/2$.

The paper is organized as follows. Throughout the paper, C denotes a positive constant and may be different in every line. Some lemmas are collected in Section 2. The proofs of the main results are given in Section 3.

2. Preliminary lemmas

In this section, we state some lemmas, which will be used in the proof of our main result. The first one comes from [12].

Lemma 2.1. *Let $\{(\mathbb{B}_k, \|\cdot\|_k), k \geq 1\}$ be a sequence of complete separable metric spaces. Let $\{X_k, k \geq 1\}$ be a sequence random variables with values in $\mathbb{B}_k$ and let $\{\Im_k, k \geq 1\}$ be a sequence of σ-fields such that X_k is $\Im_k$-measurable. Suppose that for some $\phi_k \geq 0$*

$$|\mathsf{P}(AB) - \mathsf{P}(A)\mathsf{P}(B)| \leq \phi_k \mathsf{P}(A)$$

for all $A \in \bigvee_{j<k} \Im_j$ and $B \in \Im_k$. Then, without changing its distribution we can redefine the sequence $\{X_k, k \geq 1\}$ on a richer probability space together with a sequence $\{Y_k, k \geq 1\}$ of independent random variables such that Y_k has the same distribution as X_k and

$$\mathsf{P}(\|(X_k, Y_k)\|_k \geq 6\phi_k) \leq 6\phi_k \quad k = 1, 2, \ldots.$$