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# Mohr-cyclides, a 3D representation of geological tensors: The examples of stress and flow

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**Abstract**

Mohr-circles are commonly used to represent second-rank tensors in two dimensions. In geology, this mainly applies to stress, flow, strain and deformation. Three-dimensional second rank tensors have been represented by sets of three Mohr-circles, mainly in the application of stress. This paper demonstrates that three-dimensional second rank tensors can in fact be represented in a three-dimensional reference frame by Mohr surfaces, which are members of the cyclide family. Such Mohr-cyclides can be used to represent any second rank tensor and are exemplified with the stress and flow tensors.


*Keywords:* Mohr circle; Mohr cyclide; Stress; Flow; Kinematics; Tensor

## 1. Introduction

### 1.1. Historical background

Mohr diagrams, one of the most used and useful tools in structural geology, were introduced by German scientist Otto Mohr (1882). As a civil engineer, Mohr was especially interested in mechanical forces acting on planes and, thus, presented the scientific community with a graphical representation for three-dimensional stress, plotting normal stress ($\sigma_n$) versus shear stress ($\tau$). The result was the familiar Mohr diagram for stress, consisting of the three principal circles of stress and the surface they encompass, where any plane $P$ can be plotted and assigned values for $\sigma_n$ and $\tau$, with their orientation given in terms of single or double angles. This graphical representation has since been used extensively in empirical mechanical problems, either using failure envelopes or as a tool to study fracture opening and reactivation (e.g. Delaney et al., 1986; Jolly and Sanderson, 1997).

The Mohr-circle concept was adapted for strain tensors by Nadai (1950), who devised a graphical representation of quadratic elongation versus shear strain, where angles between lines are plotted in the unstrained form. The plot is in all ways similar to Mohr's diagram and establishes a parallel between the principal circles of stress and the principal sections of the deformation ellipse. Nadai (1950) also defined a Mohr diagram for reciprocal strain, with reciprocal quadratic elongation versus reciprocal shear strain.

Mohr diagrams were formally introduced to structural geology by Brace (1961), who coined the term and explored its multiple applications in the study of deformed rocks. This new line of research was not lost and Ramsay (1967) further demonstrated the relevance of Mohr diagrams in strain analysis, showing that Mohr circles for reciprocal strain could be used to represent strain ellipses. Means (1982) introduced the Mohr diagram for the stretch tensor, where he explored the potential of polar coordinates and its applications to the study of material line behaviour, encompassing both rotational characteristics and stretch. Further research developed numerous applications of Mohr diagrams for strain to structural geology problems, namely inhomogeneous deformation (Means, 1983), strain refraction (Means, 1983; Treagus, 1983), strain

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analysis (Treagus, 1986 (which includes a comprehensive background on the history of Mohr diagrams in Structural Geology); Lisle and Ragan, 1988; Passchier, 1990a; Treagus, 1990; Simpson and De Paor, 1993; Vissers, 1994; Zhang and Zheng, 1997) and vorticity analysis (Passchier and Urai, 1988; Passchier, 1990b). Mohr diagrams for flow (velocity gradient) tensors were introduced by Lister and Williams (1983), following an idea of J.P. Platt. Since then, works like Means (1983), Bobyarchick (1986), Passchier (1986, 1987, 1988, 1993), Wallis (1992), Simpson and De Paor (1993) showed how these diagrams could be used to interpret and understand the principles of progressive deformation.

### 1.2. Tensors and Mohr-circles

As demonstrated first by Otto Mohr (1882), all tensors can be represented by Mohr diagrams. The relationship between a tensor $\mathbf{T_{ij}}$ and its Mohr-circle can be illustrated with a second-rank tensor, which requires four components (Fig. 1). In a 2D Mohr space, the vertical axis $\mathbf{T_{ij}}$ is used to plot tensor components $T_{12}$ and $T_{21}$, whereas horizontal coordinates stand for the $T_{ii}$ components, $T_{11}$ and $T_{22}$. Thus, two points can be plotted (Means, 1982): $x_1$ as ($T_{11}$, $-T_{21}$) and $x_2$ as ($T_{22}$, $T_{12}$). Either the $T_{12}$ or the $T_{21}$ sign has to be changed from the original tensor components to insure equivalence of positions above or below the horizontal axis of the Mohr diagram. The convention of Means (1982) considers $-T_{21}$, and defines Mohr-diagrams of the *first kind* (De Paor and Means, 1984). If, on the other hand, one considers $-T_{12}$, the Mohr-diagram is said to be of the *second kind* (De Paor and Means, 1984). Points $x_1$ and $x_2$ define a diameter (dashed line) of a circle, which represents the Mohr-circle of tensor $\mathbf{T_{ij}}$ (Fig. 1). Any given tensor can be described by an infinite number of sets of $\mathbf{T_{ij}}$ components, each representing a description of the tensor in a specific reference frame. Considering all these possible sets, a Mohr circle can be defined as "(...) *the geometrical locus of all possible sets of tensor components*" (Means, 1992).

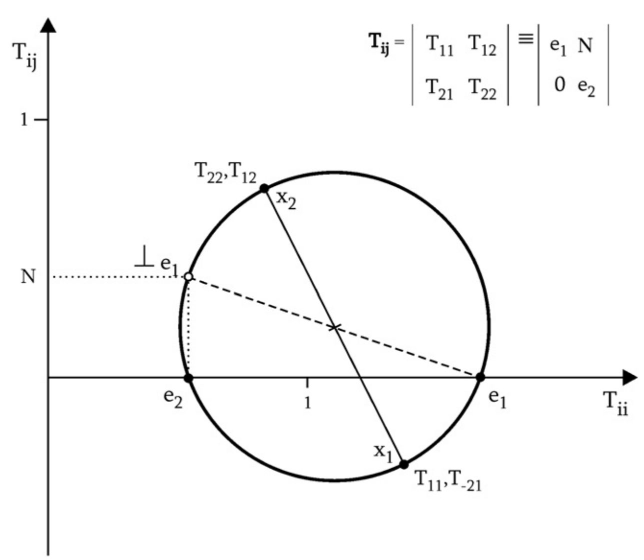


Fig. 1. Mohr-circle for an unspecified tensor $\mathbf{T_{ij}}$, defined by two alternative diameters: solid: using random tensor components; dashed: using the eigenvalues $e_1$ and $e_2$.

Second-rank tensors in three dimensions, with nine components, can also be represented by Mohr-circles. The easiest way to do this is to consider only part of the full tensor. An example of this "technique" is the literature published on velocity gradient tensors, which, for Mohr-diagram purposes, simplifies flow to monoclinic geometries, characterised by the vorticity vector parallel to one of the eigenvectors and one of the instantaneous stretching axes. Assuming this, a tensor

$$T_{ij} = \begin{vmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{vmatrix}$$

can be reduced to

$$T_{ij} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix}$$

and plotted straightforward as a Mohr-circle, ignoring the three-dimensional component given by $T_{33}$. A second method was suggested by Otto Mohr himself, for the case of stress, applied later to quadratic deformation. The stress (deformation) tensor is written as a diagonal matrix, where $T_{ii}$ are the eigenvectors of the tensor and the principal stresses $\sigma_1$,$\sigma_2$,$\sigma_3$ (for instance Fig. 3), or the principal quadratic elongations $\lambda_1$,$\lambda_2$,$\lambda_3$. These components are then used to draw three circles, or half-circles, that represent the principal sections of the stress or finite strain ellipsoid.

### 1.3. Scope

However ingenious, Mohr-circles for second-rank tensors remain simplifications because Mohr-space is always considered to be two-dimensional. This means that in order to accommodate a three-dimensional second-rank tensor in Mohr-space, it must be partitioned into three two-dimensional second-rank tensors, resulting in a combination of three Mohr-circles. In other words, the so called "three-dimensional diagram" for stress is, in fact, a two-dimensional representation of three eigenvector sections of a second-rank symmetric tensor.

The purpose of this paper is to investigate the possibility of expanding the representation of tensors into a three-dimensional Mohr-space, using examples of stress and flow. After some initial testing, it turned out that real three-dimensional Mohr-diagrams do exist and are represented by surfaces of the cyclide family and related toroids. These surfaces share all the useful properties of 2D Mohr-diagrams, with the advantage of a full three-dimensional geometry. They will be henceforth referred to as *Mohr-cyclides*. Symbols and conventions are listed in Appendix A.