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The Grünwald–Letnikov method for fractional differential equations

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ABSTRACT

This paper is devoted to the numerical treatment of fractional differential equations. Based on the Grünwald–Letnikov definition of fractional derivatives, finite difference schemes for the approximation of the solution are discussed. The main properties of these explicit and implicit methods concerning the stability, the convergence and the error behavior are studied related to linear test equations. The asymptotic stability and the absolute stability of these methods are proved. Error representations and estimates for the truncation, propagation and global error are derived. Numerical experiments are given.

1. Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past three decades mainly due to its attractive applications in numerous, seemingly diverse and wide spread fields of science and engineering. Fractional differential equations (FDEs) have been used for mathematical modeling in potential fields, hydraulics of dams, diffusion problems, waves in liquids and gases, in heat equations, especially modeling oil strata, and in Maxwell's equation. Modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives. The fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. Valuable monographs have been published on fractional calculus and applications (e.g., [1–7]). Further theoretical results, applications and modeling for ordinary and partial FDEs were discussed (e.g., [8–10]).

Numerical methods must conserve these properties of FDEs, and various schemes were proposed (e.g., [11–22]). This paper addresses the Grünwald–Letnikov approximation and follows the statement in the monography of Podlubny [6]. In a sense the resulting scheme for FDEs of fractional order α is an extension of the Euler scheme for ordinary differential equations and the given results reduce to that case if $\alpha = 1$. Consider the initial value problem

$$y'(t) = f(y(t)), \qquad y(t_0) = y_0,$$

then the explicit or implicit Euler method scheme reads

$$y_{n+1} = y_n + hf(y_n) \quad \text{or} \quad + hf(y_{n+1}).$$

Consider the initial value problem of an FDE

$$D^\alpha y(t) = f(y(t)), \qquad y(t_0) = y_0,$$

where always the existence and uniqueness of a solution is assumed. $D^\alpha y(t)$ means the derivative of order α of the function $y(t)$. The exact definition of the Riemann–Liouville and Caputo derivative is given later. There is a small difference between

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the Riemann–Liouville and the Caputo derivative. But the Caputo definition has advantages for initial value problems. M. Caputo was the first to give application of fractional calculus to mechanics, especially to linear models of viscoelasticity [23,24]. This paper mainly deals with FDEs using the Caputo operator of fractional order α with $0 < \alpha < 1$. And the Caputo derivative is approximated by the Grünwald–Letnikov approach using finite differences of fractional order. In the case of FDEs with inhomogeneous initial values, a correction term $r^{\alpha}_{n+1}y_0$ has to be added. Then the Grünwald–Letnikov scheme reads in the explicit and implicit cases as

$$y_{n+1} = c^{\alpha}_1 y_n + c^{\alpha}_2 y_{n-1} + \cdots + c^{\alpha}_{n+1} y_0 + r^{\alpha}_{n+1} y_0 + h^{\alpha} f(y_n) \quad \text{or} \quad + h^{\alpha} f(y_{n+1}).$$

The Grünwald–Letnikov method is proceeding iteratively but the sum in the scheme becomes longer and longer, which reflects the memory effect. The coefficients c^{α}_{ν} are recursively defined and show very smooth properties, e.g., they are positive and show strong damping effect. Therefore, they imply smooth properties for the scheme, but the correction term causes some perturbation. A discrete version of the Gronwall lemma applied in proofs is very useful. The properties of the Grünwald–Letnikov approximation as a numerical scheme concerning the stability and error estimates related to linear test equations are studied. Because of the long sum there arise discrepancies compared with the Euler method. One has to distinguish between the individual schemes for computing y_k and their behavior when $h \to 0$ and k is fixed, and the scheme for computing y_{n+1} at the point $t = (n+1)h$ and its behavior when $h \to 0$ and $n \to \infty$. The truncation error at the point $t = (n+1)h$ satisfies $O(h^{1+\alpha})$. But the truncation error of the scheme in the first step tends to a constant when $h \to 0$ in the case of an inhomogeneous initial value and satisfies $O(h^{\alpha})$ in the case of a homogeneous initial value. The global error is estimated by the sum of the truncation errors over all previous steps provided with damping coefficients. We can expect the order of convergence to be one. The maximum value of the global error over the whole grid is dominated by the truncation error in the first step.

Interesting examples of FDEs using the Caputo definition, denoted by D^{α}_{*}, are the Bagley–Torvik equation [25,26], [6, pp. 224–231]

$$ay''(t) + bD^{3/2}_{*} y(t) + cy(t) = f(t), \qquad y(0) = y'(0) = 0,$$

a prototype of fractional differential equations, which can be reduced to a system of FDEs of order $\alpha = 1/2$ with four equations, and further, the test equation

$$D^{\alpha}_{*} y(t) = \lambda y(t), \qquad y(0) = y_0,$$

and the fractional extension of the heat equation as a model for oil strata [27,8,28]

$$D^{\alpha}_{*} u = c^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < r, z, t < \infty, \; 0 < \alpha \le 1,$$

subject to nonstandard boundary conditions.

In Section 2, we present the definition of fractional derivatives in the sense of Riemann–Liouville and Caputo and mention the Mittag-Leffler function. The definition of fractional derivatives due to Grünwald–Letnikov is also given. The fractional order binomial coefficients and coefficients relevant for error representations are studied and monotony properties are derived in Section 3. The behavior and properties of these coefficients are investigated and explained in tables. In Section 4, the Grünwald–Letnikov scheme based on finite differences is discussed. In some sense it is an extension of the classical explicit and implicit Euler methods. The paper mainly deals with FDEs using the Caputo operator of order α. In the case of FDEs with inhomogeneous initial values, a correction term has to be added. The stability of the Grünwald–Letnikov scheme is investigated in Section 5. The asymptotic stability and absolute stability are proved. If $\alpha \to 1$ the results reduce to those of the classical Euler methods. Section 6 deals with the application of the Grünwald–Letnikov scheme to a test equation and a detailed error analysis. The error coefficients are studied when the steps are increasing. The representation of the propagation error emphasizes the stability of the Grünwald–Letnikov method caused by the strong damping factors in the fractional binomial evaluation. The global error is estimated by the sum of the truncation error of the current step and all previous steps, where again the damping effect of the fractional binomial coefficients is given. Numerical experiments are given to illustrate the properties of the schemes in the last section.

2. Fractional derivatives

There exist different approaches to fractional derivatives [6]. For simplification we consider the interval $[0, t]$ instead of $[a, t]$ and omit $a = 0$ as an index in the differential operator. Suppose that the function $y(\tau)$ satisfies some smoothness conditions in every finite interval $(0, t)$ with $t \le T$. The *Riemann–Liouville* definition ($\approx$1850) [6, p. 68] reads

$$D^{\alpha}_{R} y(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} \displaystyle\int_0^t \dfrac{y(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau & m-1 \le \alpha < m \\ \dfrac{\mathrm{d}^m y(t)}{\mathrm{d}t^m} & \alpha = m, \end{cases} \tag{2.1}$$