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On fractional integro-differential equations with state-dependent delay

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ABSTRACT

In this paper we provide sufficient conditions for the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay. A concrete application in the theory of heat conduction in materials with memory is also given.

1. Introduction

In the last two decades, the theory of fractional calculus has gained importance and popularity, due to its wide range of applications in varied fields of sciences and engineering. In [1–8] applications are mentioned to fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics and signal processing, and so on. Likewise, the functional differential equations with state-dependent delay appear frequently in applications as mathematical models, and thus have been studied extensively in the last few years (see [9–28]). However, most of the works on these equations are restricted only for the ordinary differential equations. The main object of this paper is to provide sufficient conditions for the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay of the form

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)\mathrm{d}s + f(t, u_{\rho(t,u_t)}), \quad t \in [0, b], \tag{1.1}$$

$$u(0) = \varphi \in \mathcal{B}, \tag{1.2}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, the history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, and $f : [0, b] \times \mathcal{B} \to X$ and $\rho : [0, b] \times \mathcal{B} \to (-\infty, b]$ are appropriated functions. Notice that the convolution integral in (1.1) is known as the Riemann–Liouville fractional integral.

Eqs. (1.1)–(1.2) is an abstract version of the following fractional integro-differential equation which has many physical applications, e.g., in the theory of heat conduction in materials with memory (see [29]):

$$u'(t,\xi) = \int_0^t \left(\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) u_{\xi\xi}(s,\xi)\mathrm{d}s + \left[m(t) \left(\int_0^t u(t-\sigma(\|u(t)\|), \xi')\mathrm{d}\xi' \right)^{\beta} \right], \tag{1.3}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \tag{1.4}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \leq 0,\ 0 \leq \xi \leq \pi, \tag{1.5}$$

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where $t \in [0, b]$, $\xi \in [0, \pi]$, $0 < \beta < 1 < \alpha < 2$ and $\varphi \in C_0 \times L^2(g, X)$ (see Section 4). In the literature problem (1.1)–(1.2) has been studied by several authors without delay or with delay depending only on time. In [30] the authors investigated existence and uniqueness of S-asymptotically ω-periodic mild solutions of (1.1)–(1.2) with infinite delay, while the case without delay has been considered in [31–33] (see also [34]) for existence of asymptotically almost periodic mild solutions, asymptotically behavior of solutions and existence of S-asymptotically ω-periodic mild solutions, respectively. The existence of mild solutions for the class of fractional integro-differential equations with state-dependent delay of the form (1.1)–(1.2) seems to be an unread topic.

The plan of this paper is as follows. The second section provides the necessary definitions and preliminary results. In particular, we review some of the standard properties of the solution operator generated by a sectorial operator (see Proposition 2.1). We also employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [35]. In the third section, we use fixed point theory to establish the existence of mild solutions for the problem (1.1)–(1.2). To show how easily our existence theory can be used in practice, in the fourth section we illustrate an example.

2. Preliminaries

Let $(Z, \|\cdot\|)$ and $(W, \|\cdot\|)$ be two Banach spaces. The notation $\mathcal{L}(Z, W)$ stands for the space of bounded linear operators from Z into W endowed with the uniform operator topology, and we abbreviate it to $\mathcal{L}(Z)$ whenever $Z = W$. In order to give an operator theoretical approach we recall the following definition (cf. [30]).

Definition 2.1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We recall that A is the generator of a solution operator if there exist $\mu \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X)$ such that

$$\{\lambda^\alpha : Re\lambda > \mu\} \subset \rho(A)$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t} S_\alpha(t)x\mathrm{d}t, \qquad Re\lambda > \mu, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

The concept of a solution operator, as defined above, is closely related to the concept of a resolvent family (see [29, Chapter I]). For the scalar case, there is a large bibliography, and we refer the reader to the monograph by Gripenberg et al. [36], and references therein. Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$ the family $S_\alpha(t)$ corresponds to a C_0-semigroup, whereas in the case $\alpha = 2$ a solution operator corresponds to the concept of a cosine family; see [37,38]. We note that solution operators, as well as resolvent families, are a particular case of (a, k)-regularized families introduced in [39]. According to [39] a solution operator $S_\alpha(t)$ corresponds to a $(1, \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized family. The following result is a direct consequence of [39, Proposition 3.1 and Lemma 2.2].

Proposition 2.1. *Let $S_\alpha(t)$ be a solution operator on X with generator A. Then, we have*

(a) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*

(b) *Let $x \in D(A)$ and $t \geq 0$. Then $S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(s)\mathrm{d}s$.*

(c) *Let $x \in X$ and $t \geq 0$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x\mathrm{d}s \in D(A)$ and*

$$S_\alpha(t)x = x + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x\mathrm{d}s.$$

A characterization of generators of solution operators, analogous to the Hille–Yosida Theorem for C_0-semigroup, can be directly deduced from [39, Theorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be deduced from the more general context of (a, k)-regularized resolvents (see [39–42]).

A closed and linear operator A is said to be sectorial of type μ if there exist $0 < \theta < \pi/2$, $\tilde{M} > 0$ and $\mu \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\mu + S_\theta := \{\mu + s : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\}$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{\tilde{M}}{|\lambda - \mu|}, \quad \lambda \notin \mu + S_\theta.$$

Sectorial operator are well studied in the literature. For a recent work including several examples and properties, we refer the reader to [43]. In this work we will assume that in the problem (1.1)–(1.2) the operator A is sectorial of type μ with $0 \leq \theta < \pi(1 - \alpha/2)$. Then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\mathrm{d}\lambda,$$