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Stochastic differential equations driven by a Wiener process and fractional Brownian motion: Convergence in Besov space with respect to a parameter

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ABSTRACT

A stochastic differential equation involving both a Wiener process and fractional Brownian motion, with nonhomogeneous coefficients and random initial condition, is considered. The coefficients and initial condition depend on a parameter. The assumptions on the coefficients and the initial condition supplying continuous dependence of the solution on a parameter, with respect to the Besov space norm, are established.

1. Introduction

We consider a model driven by a Wiener process and fractional Brownian motion simultaneously. On one hand, such a model includes a Wiener process that represents randomness in the sense of a lack of memory.

On the other hand, fractional Brownian motion provides a non-Markov component. It so happens that processes in hydrodynamics, telecommunications, economics, and finance demonstrate the availability of random noise that can be modeled by a Wiener process and also a so called long memory that can be modeled with the help of fractional Brownian motion with Hurst index $H > \frac{1}{2}$. We consider stochastic differential equations depending on a parameter u, with nonhomogeneous coefficients, defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0, T]), \mathbb{P})$:

$$X_t^u = X_0^u + \int_0^t a^u(s, X_s^u)\mathrm{d}s + \int_0^t b^u(s, X_s^u)\mathrm{d}W_s + \int_0^t c^u(s, X_s^u)\mathrm{d}B_s^H, \quad t \in [0, T], \tag{1}$$

where X_0^u is an $\mathcal{F}_0$-measurable random variable, $\mathbb{E}(X_0^u)^2 < \infty$ for all $u \in [0, u_0]$, $W = (W_t, \mathcal{F}_t, t \in [0, T])$ is a Wiener process, and $B^H = (B_t^H, \mathcal{F}_t, t \in [0, T])$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. The coefficients $a^u, b^u, c^u : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are measurable deterministic functions, satisfying for each $u \in [0, u_0]$ the conditions of existence and uniqueness of the solution of (1) (see, e.g., [1,2]). Our aim is to establish the assumptions on the coefficients and on the initial condition supplying the convergence of X^u in Besov space to the solution of the limit equation.

The paper is organized as follows. In Section 2 we revise some results concerning fractional calculus including pathwise stochastic integration with respect to fractional Brownian motion and some estimates for such integrals. Section 3 contains the theorem concerning the existence and uniqueness of the solution of the mixed stochastic differential equation (1); for the proof of this result, see [2]. In Section 4 we prove that under uniform bounds on the coefficients, an a.s. uniform estimate for the norm of solution (1) exists. In Section 5 the convergence in Besov space of the solutions depending on a parameter is established. Section 6 contains two examples, and Section 7 concludes.

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2. Elements of fractional calculus

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0, T]), \mathbb{P})$ be a complete probability space with a filtration satisfying the standard conditions. Denote by $\{W_t, \mathcal{F}_t, t \in [0, T]\}$ the standard Wiener process adapted to this filtration.

Definition 2.1. The fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t, t \geq 0\}$, having the properties $B_0^H = 0$, $\mathbb{E}B_t^H = 0$, and $\mathbb{E}B_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.

Remark 2.1. Fractional Brownian motion has a continuous modification, according to the Kolmogorov theorem. In what follows we consider this continuous modification. Also, we suppose that our fBm is adapted to the filtration $\{\mathcal{F}_t, t \in [0, T]\}$. (We can suppose that $\mathcal{F}_t, t \in [0, T]$, is generated by W and B^H.)

To construct the integral with respect to fractional Brownian motion, we use the generalized (fractional) Lebesgue–Stieltjes integral (see [3–5]). In order to introduce it, consider two nonrandom functions f and g, defined on some interval $[a, b] \subset R$. Suppose also that the limits $f(u+) := \lim_{\delta \downarrow 0} f(u+\delta)$ and $g(u-) := \lim_{\delta \downarrow 0} g(u-\delta)$, $a \leq u \leq b$, exist. Put $f_{a+}(x) := (f(x) - f(a+))1_{(a,b)}(x)$, $g_{b-}(x) := (g(b-) - g(x))1_{(a,b)}(x)$. Suppose that $f_{a+} \in I_{a+}^{\alpha}(L_p[a, b])$, $g_{b-} \in I_{b-}^{1-\alpha}(L_q[a, b])$, for some $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$. (For the notation and the statements concerning fractional analysis, see [6].) Consider the fractional derivatives

$$(D_{a+}^{\alpha} f_{a+})(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f_{a+}(x)}{(x-a)^{\alpha}} + \alpha \int_{\alpha}^{x} \frac{f_{a+}(x) - f_{a+}(u)}{(x-u)^{1+\alpha}} \mathrm{d}u \right) 1_{(a,b)}(x),$$

$$(D_{b-}^{1-\alpha} g_{b-})(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g_{b-}(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_{x}^{b} \frac{g_{b-}(x) - g_{b-}(u)}{(u-x)^{2-\alpha}} \mathrm{d}u \right) 1_{(a,b)}(x).$$

Note that $D_{a+}^{\alpha} f_{a+} \in L_p[a, b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a, b]$.

Definition 2.2 (*[4,5]*)**.** Under the above assumptions, the generalized (fractional) Lebesgue–Stieltjes integral $\int_a^b f(x)\mathrm{d}g(x)$ is defined as

$$\int_a^b f(x)\mathrm{d}g(x) := \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)\mathrm{d}x + f(a+)(g(b-) - g(a+)),$$

and for $\alpha p < 1$ it can be simplified to

$$\int_a^b f(x)\mathrm{d}g(x) := \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)\mathrm{d}x. \tag{2}$$

As follows from [6], for any $1 - H < \alpha < 1$ there exists a fractional derivative $D_{b-}^{1-\alpha} B_{b-}^H(x) \in L_{\infty}[a, b]$. Therefore, for $f \in I_{a+}^{\alpha}(L_1[a, b])$ we can define the integral w.r.t. the fBm according to (2).

Definition 2.3. The integral with respect to the fBm is defined as

$$\int_a^b f\mathrm{d}B^H := \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B_{b-}^H)(x)\mathrm{d}x.$$

Consider the fixed interval $[0, T]$ and $\alpha \in (1-H, 1/2)$. We also denote by $W^{\alpha,1}(0, T)$ the space of measurable functions f on $[0, T]$ such that

$$\|f\|_{\alpha,0,T} := \int_0^T \frac{|f(s)|}{s^{\alpha}} \mathrm{d}s + \int_0^T \int_0^s \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} \mathrm{d}r\mathrm{d}s < \infty. \tag{3}$$

The stochastic integral with respect to the fBm from Definition 2.3 admits the following estimate: for any $\alpha \in (1-H, 1/2)$ and any $t \in (0, T)$ there exists a random variable $\psi(\omega, \alpha, t)$ with moments of any order such that for any function $f \in W^{\alpha,1}(0, T)$ we have

$$\begin{aligned} \left| \int_0^t f(s)\mathrm{d}B_s^H \right| &\leq \psi(\omega, \alpha, t) \left(\int_0^t \frac{|f(s)|}{s^{\alpha}} \mathrm{d}s + \int_0^t \int_0^s \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} \mathrm{d}r\mathrm{d}s \right) \\ &=: \psi(\omega, \alpha, t)\|f\|_{\alpha,0,t}. \end{aligned} \tag{4}$$

Also we need the following corollary from Proposition 4.1 in [3]: for any $\alpha \in (1-H, 1/2)$, for all $0 \leq s < t \leq T$ and for any function $f \in W^{\alpha,1}(0, T)$, we have

$$\left| \int_s^t f(s)\mathrm{d}B_s^H \right| \leq \psi(\omega, \alpha, t) \left(\int_s^t \frac{|f(r)|}{(r-s)^{\alpha}} \mathrm{d}s + \int_s^t \int_s^r \frac{|f(r) - f(v)|}{(r-v)^{\alpha+1}} \mathrm{d}v\mathrm{d}r \right). \tag{5}$$