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Existence results for fractional neutral integro-differential equations with state-dependent delay

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ABSTRACT

In this paper we study the existence of mild solutions for a class of abstract fractional neutral integro-differential equations with state-dependent delay. The results are obtained by using the Leray–Schauder alternative fixed point theorem. An example is provided to illustrate the main results.

1. Introduction

In this paper we study the existence of mild solutions for a class of partial neutral integro-differential equations with state-dependent delay described in the abstract form

$$D_t^\alpha N(x_t) = AN(x_t) + \int_0^t B(t-s)N(x_s)\mathrm{d}s + f(t, x_{\rho(t,x_t)}), \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \qquad x'(0) = 0, \tag{1.2}$$

where $\alpha \in (1, 2)$, $t \in I = [0, b]$, A, $(B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space $(X, \|\cdot\|)$, and $D_t^\alpha h(t)$ represents the Caputo derivative of order $\alpha > 0$ defined by

$$D_t^\alpha h(t) := \int_0^t g_{n-\alpha}(t-s)\frac{\mathrm{d}^n}{\mathrm{d}s^n}h(s)\mathrm{d}s,$$

where n is the smallest integer greater than or equal to α and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta \geq 0$. The history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f, g : [0, b] \times \mathcal{B} \to X$, $N(\psi) = \psi(0) + g(t, \psi)$, $\psi \in \mathcal{B}$ and $\rho : [0, b] \times \mathcal{B} \to (-\infty, b]$ are appropriate functions.

Functional differential equations with state-dependent delay appear frequently in applications as model equations and for this reason the study of such equations has received great attention in the last few years. The literature devoted to this subject is concerned fundamentally with first-order functional differential equations for which the state belongs to some finite dimensional space; see, among another works, [1–9]. The problem of the existence of solutions for partial functional differential equations and partial neutral functional differential equations with state-dependent delay has been recently treated in the literature in [10–15].

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We observe that the fractional order can be complex from the viewpoint of pure mathematics and there is much interest in developing the theoretical analysis and numerical methods of fractional equations, because they have recently proved to be valuable in various fields of sciences and engineering [16–18]. For details, including some applications and recent results, see the monographs of Ahn and McVinish [19], Gorenglo and Mainardi [20], Hilfer [21], Miller and Ross [22], and the papers of Agarwal et al. [23–29], Benchohra et al. [30], Cuevas et al. [31–34], Lakshmikantham et al. [35,36] and Yong Zhou et al. [37–39] (see also [40–43] and references therein). Our purpose in this paper is to establish the existence of mild solutions for a fractional neutral integro-differential equations with state-dependent delay.

2. Preliminaries

In what follows we recall some definitions, notation and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and A, $B(t)$, $t \geq 0$, are closed linear operators defined on a common domain $\mathcal{D} = D(A)$ which is dense in X. The notation $[D(A)]$ represents the domain of A endowed with the graph norm. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ the notation $\widehat{K}$ denotes the Laplace transform of K. The notation, $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. On the other hand, for a bounded function $\gamma : [0, a] \to Z$ and $t \in [0, a]$, the notation $\|\gamma\|_{Z,t}$ is defined by

$$\|\gamma\|_{Z,t} = \sup\{\|\gamma(s)\|_Z : s \in [0, t]\}, \tag{2.3}$$

and we simplify this notation to $\|\gamma\|_t$ when no confusion about the space Z arises.

To obtain our results, we assume that the abstract fractional integro-differential problem

$$D_t^{\alpha} x(t) = Ax(t) + \int_0^t B(t-s)x(s)\mathrm{d}s, \tag{2.4}$$

$$x(0) = z \in X, \qquad x'(0) = 0, \tag{2.5}$$

has an associated α-resolvent operator of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X.

Definition 2.1. A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X is called an α-resolvent operator of (2.4)–(2.5) if the following conditions are verified.

(a) The function $\mathcal{R}_\alpha(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in X$ and $\alpha \in (1, 2)$.
(b) For $x \in D(A)$, $\mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(A)]) \bigcap C^1((0, \infty), X)$, and

$$D_t^{\alpha}\mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t B(t-s)\mathcal{R}_\alpha(s)x\mathrm{d}s, \tag{2.6}$$

$$D_t^{\alpha}\mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)B(s)x\mathrm{d}s, \tag{2.7}$$

for every $t \geq 0$.

The existence of an α-resolvent operator for problem (2.4)–(2.5) was studied in [25]. In this work we have considered the following conditions.

(P1) The operator $A : D(A) \subseteq X \to X$ is a closed linear operator with $[D(A)]$ dense in X. Let $\alpha \in (1, 2)$. For some $\phi_0 \in (0, \frac{\pi}{2}]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each

$$\lambda \in \Sigma_{0,\alpha\vartheta} = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |arg(\lambda)| < \alpha\vartheta\},$$

where $\vartheta = \phi + \frac{\pi}{2}$ and $\|R(\lambda, A)\| \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\alpha\vartheta}$.

(P2) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Sigma_{0,\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\Sigma_{0,\vartheta}$ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\|\,\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$.

(P3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant C_1 such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, and $\|A\widehat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in D$ and all $\lambda \in \Sigma_{0,\vartheta}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \vartheta)$,

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\lambda| > r,\ |arg(\lambda)| < \theta\},$$