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Existence and uniqueness of mild solutions for abstract delay fractional differential equations

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ABSTRACT

In this paper, by means of solution operator approach and contraction mapping theorem, the existence and uniqueness of mild solutions for a class of abstract delay fractional differential equations are obtained.

1. Introduction

We are concerned with the existence and uniqueness of mild solutions for the following abstract delay fractional differential equations

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + J_t^{1-\alpha} f(t, u_t), & \text{for } t \in [0, T], \\ u(t) = \varphi(t), & \text{for } t \in [-r, 0], \end{cases} \tag{1}$$

where $\alpha \in (0, 1)$, ${}^C D_t^\alpha$ is the Caputo fractional derivative operator of order α, $J_t^{1-\alpha}$ is the Riemann–Liouville fractional integral operator of order $1-\alpha$, $A : D(A) \subset X \to X$ is the infinitesimal generator of a solution operator $\{S(t)\}_{t\geq 0}$, $D(A)$ is the domain of A equipped with the graph norm, X is a Banach space, $f : [0, T] \times C([-r, 0]; X) \to X$ is a continuous function. $r, T > 0$ are given positive real numbers. $C([-r, 0]; X)$ denotes the space of continuous functions from $[-r, 0]$ to X equipped with the sup-norm. For $u \in C([-r, T]; X)$ and $t \in [0, T]$, let u_t denote the element of $C([-r, 0]; X)$ defined by $u_t(\theta) = u(t + \theta)$, $-r \leq \theta \leq 0$.

Fractional derivatives describe the property of memory and heredity of materials, and it is the major advantage of fractional derivatives compared with integer order derivatives. For more details about fractional calculus and fractional differential equations we refer to the books by Podlubny [1], Sabatier et al. [2] and the papers by Baeumer et al. [3], Orsingher and Beghin [4], Lu and Chen [5], Eidelman and Kochubei [6], Anh and Leonenko [7], Nigmatullin [8], Delbosco and Rodino [9], Zhou and Jiao [10], Wang and Zhou [11] and Zhou et al. [12,13]. Integer order derivatives (integrals) can be seen as the limits of fractional order derivatives (integrals) under national conditions on some function $h(t)$. In fact, suppose $h : (0, \infty) \to (-\infty, \infty)$ is a real function, let $\alpha \to 1-$, we obtain ${}^C D_t^\alpha h(t) \to \frac{\mathrm{d}}{\mathrm{d}t} h(t)$ and $J_t^{1-\alpha} h(t) \to h(t)$. If ${}^C D_t^\alpha$ is replaced by the first differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$, and $J_t^{1-\alpha}$ is replaced by the identity operator we have the following version of (1)

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t, u_t), & \text{for } t \in [0, T], \\ u(t) = \varphi(t), & \text{for } t \in [-r, 0]. \end{cases} \tag{2}$$

It is well known that (see e.g., [14]) the semigroup theory ensures the well-posedness of problem (2) when A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (C_0 semigroups).

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In some publications, the authors note that the concepts of mild solutions for Cauchy problems with Caputo derivatives are not appropriate. For example, Jaradat et al. [15] defined a continuous solution $u(t)$ of the integral equations

$$u(t) = T(t-t_0)u_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T(t-s)f(s,u(s),Gu(s),Su(s))\mathrm{d}s$$

as a mild solution of the initial value problem

$$\begin{cases} u^{(\alpha)}(t) = Au(t) + f(t,u(t),Gu(t),Su(t)), & t\in[t_0,T],\\ u(t_0)=u_0, \end{cases} \tag{3}$$

where $\alpha \in (0,1]$, $u^{(\alpha)}(t)$ is the Caputo derivative of order α, A is the infinitesimal generator of a strongly continuous semigroups $\{T(t);\ t\geq 0\}$ on a Banach space X.

Let us take a special case, let $X = R, A = a > 0, f \equiv 0, t_0 = 0$, then problem (3) becomes

$$\begin{cases} {}^{C}D_t^{\alpha}u(t) = au(t), & t\in[0,T],\\ u(0)=u_0. \end{cases} \tag{4}$$

It is easy to see that $u(t) = E_\alpha(at^\alpha)u_0$ is the unique continuous solution to (4), where $E_\alpha(z)$ denotes the Mittag-Leffler function, which is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

where $\Gamma(\cdot)$ is the Gamma function. It is proved in [16] that the function $E_\alpha(at^\alpha)$ cannot possess the semigroup property for $\alpha\in(0,1), a>0, t\geq 0$. Hence the concept of mild solutions in [15] is not suitable. Similar problems appear in [20–22].

Just like the semigroup theory is inevitable for the classical abstract Cauchy problem, the concept of the solution operator (or resolvent) is central for the theory of fractional evolution equations with Caputo derivatives. Bazhlekova [17] used solution operator to investigate the following fractional Cauchy problem

$${}^{C}D_t^{\alpha}u(t) = Au(t), \qquad u(0)=x; \qquad u^{(k)}(0)=0, \quad k=0,1,\ldots,m-1, \tag{5}$$

where $\alpha > 0, m = \lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α, $A : D(A) \to X$ is a densely closed linear operator. ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative operator defined by

$${}^{C}D_t^{\alpha}u(t) = D_t^{\alpha}\left(u(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}u^{(k)}(0)\right), \tag{6}$$

where D_t^{α} is the Riemann–Liouville derivative of order α.

Chen and Li [18] present the notion of the α-resolvent operator function, they proved that for $\alpha > 0$ a family $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent operator function if and only if it is the solution operator of (5).

The purpose of this paper is to study the existence and uniqueness of mild solutions for (1) by virtue of solution operator method and contraction mapping theorem.

2. Preliminaries

In this section, we recall some definitions and propositions of fractional calculus and solution operator.

Let X be a Banach space. Let $\alpha > 0,\ m = \lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α. By $C([0,T];X)$, resp. $C^m([0,T];X)$, we denote the spaces of functions $u : [0,T] \to X$, which are continuous, resp. m-times continuous differentiable function from $[0,T]$ to X. $C([0,T];X)$ and $C^m([0,T];X)$ are Banach space equipped with the norms

$$\|u\|_C = \sup_{t\in[0,T]}\|u(t)\|_X, \qquad \|u\|_{C^m} = \sup_{t\in[0,T]}\sum_{k=0}^{m}\|u^{(k)}(t)\|_X.$$

Let $I=(a,b)$, where $-\infty\leq a\leq b\leq+\infty$, $1\leq p<\infty$. $L^p(I;X)$ denotes the space of all Bochner-measurable functions $u : I\to X$ such that $\|u(t)\|_X^p$ is integrable, it is a Banach space with the norm

$$\|u\|_{L^p(I;X)} = \left(\int_I \|u(s)\|_X^p\mathrm{d}s\right)^{1/p}.$$

Let N, R denote the sets of natural, real numbers, respectively. $R_+ = [0,\infty)$. Let $J=(0,T)$ or $J=R_+$, or $J=R$, $1\leq p<\infty$, $n\in N$. The Sobolev spaces can be defined as

$$W^{n,p}(J;X) = \left\{u \mid \exists\Phi\in L^p(J;X) : u(t) = \sum_{k=0}^{n-1}c_k\frac{t^k}{k!} + \frac{t^{n-1}}{(n-1)!}*\Phi(t),\ t\in J\right\}.$$

In fact, $\Phi(t) = u^{(n)}(t)$, $c_k = u^{(k)}(0)$.