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Optimal control of a fractional diffusion equation with state constraints

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ABSTRACT

This paper is concerned with the state constrained optimal control problems of a fractional diffusion equation in a bounded domain. The fractional time derivative is considered in the Riemann–Liouville sense. Under a Slater type condition we prove the existence a Lagrange multiplier and a decoupled optimality system.

1. Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we consider the fractional diffusion equation:

$$\begin{cases} D^{\alpha}_{RL} y - \Delta y = h + v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(0^+) = 0 & \text{in } \Omega \end{cases} \tag{1}$$

where $0 < \alpha < 1$, the control v and the function h belong to $L^2(Q)$, the fractional integral $I^{1-\alpha}$ and derivative D^{α}_{RL} are understood here in the Riemann–Liouville sense, $I^{1-\alpha}y(0^+) = \lim_{t\to 0^+} I^{1-\alpha}y(t)$.

Fractional-order models seem to be more adequate than integer-order models because fractional derivatives provide an excellent tool for the description of memory and heredity effects of various materials and processes, including gas diffusion and heat conduction, in fractal porous media [1,2]. Sokolov et al. [3], proved that fractional diffusion equations generalize Fick's second law and the Fokker–Planck equation by taking into account memory effects such as the stretching of polymers under external fields and the occupation of deep traps by charge carriers in amorphous semiconductors. Oldham and Spanier [4] discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. Mainardi [5] and Mainardi et al. [6,7] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2.

Optimal control problems with integer order have been widely studied and many techniques have been developed for solving such problems [8–11]. Also, state constrained optimal control problems have attracted several authors in the last three decades, mostly for their importance in various applications in optimal control partial differential equations with an integer time derivative. For such problems, it is well-known that one can derive optimality conditions if one can prove the existence of a Lagrange multiplier associated with the constraint in the state (see for instance [12,13]). For instance, considering a quadratic control for elliptic equations with pointwise constraints, Casas [14] proved the existence of a Lagrange multiplier and derived an optimality condition using results of convex analysis. Barbu and Precupanu [9] and Lasiecka [15] derived the existence of a Lagrange multiplier for some optimal control with integral state constraints. Considering a parabolic system controlled by Neumann conditions and subject to pointwise state constraints on the final

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state, Mackenroth [16] prove the existence of a multiplier as a solution of a dual problem. By a penalization method, Bergounioux [10] and Bergounioux and Tiba [17] proved the existence of a multiplier and derived optimal conditions for elliptic and parabolic equations with state constraints respectively.

In the area of calculus of variations and optimal control of fractional differential equations, little has been done since that problem has only been recently considered. The first record of the formulation of the fractional optimal control problem was given by Agrawal in [18] where he presented a general formulation and proposed a numerical method to solve such problems. In that paper, the fractional derivative was defined in the Riemann–Liouville sense and the formulation was obtained by means of fractional variation principle [19] and the Lagrange multiplier technique. Following the same technique, Frederico et al. [20] obtained a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. Recently, Agrawal [21] presented an eigenfunction expansion approach for a class of distributed system whose dynamics are defined in the Caputo sense. Following the same approach as Agrawal, in [22] Özdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the Riemann–Liouville sense. In [23], Jelicic et al. formulated necessary conditions for optimal control problems with dynamics described by differential equations of fractional order. Using an expansion formula for the fractional derivative, they proposed optimality conditions and a new solution scheme, using an expansion formula for the fractional derivative. In [24], Baleanu et al. described a formulation for fractional optimal control problems defined in multi-dimensions when the dimensions of the state and control variables are unlike each other. The problem is formulated with the Riemann–Liouville fractional derivatives and the fractional differential equations involving the state and control variables are solved using Grünwald–Letnikov approximation. Zhou [25] considered the following Lagrange problem:

Find $(x_0, u_0) \in C([0, T], \mathbb{X}) \times U_{ad}$ solution of

$$\min_{u\in U_{ad}} \int_0^T \mathcal{L}(t, x^u(t), u(t))\mathrm{d}t$$

where $\mathbb{X}$ is a Banach space, $T > 0$, $C([0, T], \mathbb{X})$ denotes the space of all $\mathbb{X}$-value functions defined and continue on $[0, T]$ and x^u denotes the solution of system $D^\alpha x(t) = -Ax(t) + f(t, x(t)) + C(t)u(t),\ t \in [0, T]; x(0) = x_0$. Under a suitable condition on $\mathcal{L}$, he proved that the Lagrange problem has at least one optimal pair. In [26] Mophou considered the following fractional optimal control problem: find the control $u = u(x, t) \in L^2(Q)$ that minimizes the cost function

$$J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \quad z_d \in L^2(Q) \text{ and } N > 0$$

subject to the system (1) with $h \equiv 0$. The author proved that the optimal control problem has a unique solution and derived an optimality system. We also refer to [27] where boundary fractional optimal control with finite observation expressed in terms of a Riemann–Liouville integral of order α is studied.

In this paper, we are concerned with a fractional optimal control with constraints on the state. More precisely, we first prove that under the above assumptions on the data, Problem (1) has a unique solution in $L^2(0, T; H^2 \cap H^1_0(\Omega))$ (see Theorem 2.10). Then we define the affine application $\mathbb{T}$, from $L^2(Q)$ to $L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega))$ such that $y = \mathbb{T}(v)$ is the unique solution of (1). We also define the functional $J : L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(Q) \to R_+$ by

$$J(y, v) = \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)} \tag{2}$$

where $z_d \in L^2(Q)$ and $N > 0$.

Finally, we consider the following optimal control problem with constraint on the state:

$$\begin{cases} \min J(y, v), \\ y = \mathbb{T}(v), \\ y \in K \text{ and } v \in \mathcal{U}_{ad} \end{cases} \tag{3}$$

where K and $\mathcal{U}_{ad}$ are two nonempty closed convex subsets of $L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega))$ and $L^2(Q)$ respectively. Using a penalization method, we prove the existence of a Lagrange multiplier and a decoupled optimality condition for the fractional diffusion (1). To the best of our knowledge, the fractional optimal control problem (3) is new since most fractional optimal control problems in the literature are considered for a performance index subject to the system dynamic constraints and the initial condition.

The rest of the paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3 we show that our optimal control problem holds and under a Slater type condition we prove the existence of a Lagrange multiplier and a decoupled optimality system. Concluding remarks are presented in Section 4.

2. Preliminaries

Definition 2.1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}_+$ and $\alpha > 0$. Then the expression

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0$$

is called the Riemann–Liouville integral of the function f order α.