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Approximating n-time differentiable functions of selfadjoint operators in Hilbert spaces by two point Taylor type expansion

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ABSTRACT

On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some approximations for the n-time differentiable functions of selfadjoint operators in Hilbert spaces by two point Taylor type expansions are given.

1. Introduction

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann–Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\mathrm{d}E_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, \mathrm{d}\langle E_\lambda x, y\rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [1] and the references therein.

For other recent results, see [2–12].

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The following result provides a Taylor type representation for a function of selfadjoint operators in Hilbert spaces with an integral remainder:

Theorem 1 (*Dragomir, 2010, [13]*). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its* spectral family, *I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the nth derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the equalities*

$$f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(A - c1_H)^k + R_n(f, c, m, M) \tag{1.3}$$

where

$$R_n(f, c, m, M) = \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^\lambda (\lambda - t)^n \, \mathrm{d}\left(f^{(n)}(t)\right) \right) \mathrm{d}E_\lambda. \tag{1.4}$$

This representation provides the following vectorial error bounds.

Theorem 2 (*Dragomir, 2010, [13]*). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its* spectral family, *I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the nth derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$\begin{aligned}
&\left| \langle f(A)x, y\rangle - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left\langle (A - c1_H)^k x, y \right\rangle \right| \\
&\leq \frac{1}{n!} \left[(c-m)^n \bigvee_m^c \left(f^{(n)}\right) \bigvee_m^c \left(\langle E_{(\cdot)}x, y\rangle\right) + (M-c)^n \bigvee_c^M \left(f^{(n)}\right) \bigvee_c^M \left(\langle E_{(\cdot)}x, y\rangle\right) \right] \\
&\leq \frac{1}{n!} \max\left\{ (M-c)^n \bigvee_c^M \left(f^{(n)}\right), (c-m)^n \bigvee_c^M \left(f^{(n)}\right) \right\} \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\leq \frac{1}{n!} \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^n \bigvee_m^M \left(f^{(n)}\right) \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right),
\end{aligned} \tag{1.5}$$

for any $x, y \in H$.

For other error bounds in the case when the nth derivative $f^{(n)}$ is Lipschitzian and some applications for particular functions including the exponential and logarithmic function see [13].

As one can see, by choosing in (1.5) either $c = m$, $c = M$ or $c = \frac{m+M}{2}$, that one can obtain some Taylor like expansions in terms of the function and the derivative values in that specific point. The error estimation is best when c is taken in the middle of the interval $[m, M]$ where the spectrum of the operator is located.

In this paper, however we develop a Taylor type expansion in terms of the function and the derivative values in both extremal points m and M. Applications for some elementary functions of interest including the logarithmic and exponential functions are also provided.

2. Representation results

We start with the following identity that has been obtained in [14]. For the sake of completeness we give here a short proof as well.

Lemma 1. *Let I be a closed subinterval on $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let n be a nonnegative integer. If $f : I \to \mathbb{R}$ is such that the nth derivative $f^{(n)}$ is of bounded variation on the interval $[a, b]$, then, for any $x \in [a, b]$ we have the representation*

$$\begin{aligned}
f(x) = {} & \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right] + \frac{(b-x)(x-a)}{b-a} \\
& \times \sum_{k=1}^{n} \frac{1}{k!} \left\{ (x-a)^{k-1} f^{(k)}(a) + (-1)^k (b-x)^{k-1} f^{(k)}(b) \right\} + \frac{1}{b-a} \int_a^b S_n(x, t) \mathrm{d}\left(f^{(n)}(t)\right),
\end{aligned} \tag{2.1}$$

where the kernel $S_n : [a, b]^2 \to \mathbb{R}$ is given by

$$S_n(x, t) = \frac{1}{n!} \times \begin{cases} (x-t)^n (b-x) & \text{if } a \leq t \leq x; \\ (-1)^{n+1} (t-x)^n (x-a) & \text{if } x < t \leq b \end{cases} \tag{2.2}$$

and the integral in the remainder is taken in the Riemann–Stieltjes sense.