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The discrete analogue of Laplace's method

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ABSTRACT

We give a justification of the discrete analogue of Laplace's method applied to the asymptotic estimation of sums consisting of positive terms. The case considered is the series related to the hypergeometric function ${}_pF_{q-1}(x)$ (with $q \geq p+1$) as $x \to +\infty$ discussed by Stokes [G.G. Stokes, Note on the determination of arbitrary constants which appear as multipliers of semi-convergent series, Proc. Camb. Phil. Soc. 6 (1889) 362–366]. Two examples are given in which it is shown how higher order terms in the asymptotic expansion may be derived by this procedure.

1. Introduction

Laplace's approximation is one of the most fundamental asymptotic techniques for the estimation of integrals containing a large parameter or variable. For integrals of the form

$$I(x) = \int_a^b f(t) e^{x\psi(t)} dt \quad (x \to +\infty),$$

where $f(t)$ and $\psi(t)$ are real continuous functions defined on the interval $[a, b]$ (which may be infinite), the dominant contribution as $x \to +\infty$ arises from a neighbourhood of the point where $\psi(t)$ attains its maximum value. When $\psi(t)$ possesses a single maximum at the point $t_0 \in (a, b)$, so that $\psi'(t_0) = 0$, $\psi''(t_0) < 0$ and $f(t_0) \neq 0$, then $I(x)$ has the asymptotic behaviour

$$I(x) \sim f(t_0) e^{x\psi(t_0)} \left(\frac{-2\pi}{x\psi''(t_0)} \right)^{1/2} \quad (x \to +\infty);$$

see, for example, [1, p. 39] or [2, p. 57].

The same principle may also be applied to the sum of a series of positive terms, in which the terms steadily increase up to a certain point and then steadily decrease. The asymptotic behaviour of the sum of the series can then be obtained by a discrete analogue of Laplace's method by consideration of the order of magnitude of the greatest term in the series. In 1889, Stokes [3] published a short paper in which he applied this principle to obtain the leading asymptotic behaviour of the hypergeometric-type series

$$F(x) = \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} \Gamma(n+a_r)}{\prod_{r=1}^{q} \Gamma(n+b_r)} x^n \quad (q \geq p+1,\ |x| < \infty), \tag{1.1}$$

where $p \geq 0$, $q \geq 1$ are integers, a_r $(1 \leq r \leq p)$ and b_r $(1 \leq r \leq q)$ are positive parameters and $x > 0$. The function $F(x)$ covers many cases of important special functions in physical applications and, when $b_q = 1$ say, is proportional to the

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generalised hypergeometric function ${}_pF_{q-1}(x)$ with numeratorial parameters a_r $(1 \le r \le p)$ and denominatorial parameters b_r $(1 \le r \le q-1)$. Stokes argued that the dominant contribution to $F(x)$ as $x \to \infty$ arose from the terms in the series situated in the neighbourhood of the greatest term. By approximation of these terms in the form of a Gaussian exponential followed by replacement of the sum by an integral with limits extended to $\pm\infty$, Stokes showed in a non-rigorous fashion that

$$F(x) \sim (2\pi)^{(1-\kappa)/2}\kappa^{-1/2}x^{(\vartheta+\frac{1}{2})/\kappa}\exp(\kappa x^{1/\kappa}) \quad (x \to +\infty), \tag{1.2}$$

where

$$\kappa = q - p, \qquad \vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \frac{1}{2}\kappa. \tag{1.3}$$

This result appears to be the earliest attempt at the determination of the asymptotic behaviour of a series of the form (1.1). An application of this principle that the large-argument behaviour of a series is controlled by the magnitude of the greatest term was made by Hardy [4] in the determination of the zeros of a class of integral functions.

Relatively little use appears to have been made of the discrete analogue of Laplace's method, presumably on account of its being confined to a series of positive terms and the heuristic nature of its arguments. Examples can be found in [5, p. 8] and in the book by Bender & Orszag [6, p. 304], where they derive the leading asymptotic behaviour of the sum

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^\alpha} \quad (\alpha > 0) \tag{1.4}$$

as $x \to +\infty$. In the case of integer α, this latter function is a particular case of $F(x)$ in (1.1) with $p = 0$, $q = \alpha$. The same example has been discussed in Olver's book [7, p. 307] but using a different approach based on an integral representation of the sum together with Laplace's approximation for integrals.[1] A recent proof of a discrete analogue of Laplace's method applied to sums of the form $\sum_{k=0}^{n} f_n(k) q^{g_n(k)}$ as $n \to +\infty$, where $f_n(k)$ and $g_n(k)$ are functions defined on nonnegative integers and $0 < q < 1$, has been given in [8]. These authors applied their results to derive asymptotic formulae for the q^{-1}-Hermite, the Stieltjes–Wigert and the q-Laguerre polynomials.

In this paper, we present a justification of Stokes' arguments for the discrete analogue of Laplace's method applied to the function $F(x)$ in (1.1). Although other more general methods are available for the asymptotics of $F(x)$ when x is a large complex variable, namely the classical Laplace method applied to an integral representation of the sum in (1.1) or the asymptotic theory of hypergeometric-type functions developed in [9,10], our aim here is to put Stokes' arguments on a more rigorous foundation. We shall restrict our attention to the case of positive parameters and the variable x considered by Stokes, although it may be possible to extend the arguments to cover the case of complex x by using the ideas given in [11] applied to the determination of the relation of the maximum modulus of an integral function to its maximum term. In addition, we shall show how higher order terms in the expansion of $F(x)$ as $x \to +\infty$ can also be derived.

The results obtained are then used to give expansions for two functions. The first example is the sum defined in (1.4) and the second example comes from a problem in combinatorics expressed in the form of an integral of a product of Hermite polynomials over $(-\infty, \infty)$.

2. Preliminary lemmas

We first state and prove two lemmas that will be required in the asymptotic discussion of $F(x)$. In the course of our analysis it is necessary to introduce the positive parameter $\epsilon < 1$ that will be chosen to scale with the asymptotic variable x given by

$$\epsilon \sim x^{-\nu}, \quad \frac{1}{3} < \nu < \frac{1}{2}\ (x \to +\infty). \tag{2.1}$$

Lemma 1. *Let $a = \kappa/(2x)$, $0 \le \delta < 1$ and $\mu \sim \epsilon x$, where κ is defined in (1.3), $x > 0$ and ϵ is specified in (2.1). Then for nonnegative integer r, we have*

$$S_\pm \equiv \sum_{k=\mu+1}^{\infty} (k \pm \delta)^r e^{-a(k\pm\delta)^2} = O(x^{r+\nu}e^{-a\mu^2}) \quad (r = 0, 1, 2, \ldots) \tag{2.2}$$

as $x \to \infty$ $(a \to 0+)$.

Proof. Consider first the sum

$$S_+ = \sum_{k=\mu+1}^{\infty} (k+\delta)^r e^{-a(k+\delta)^2} < \sum_{k=\mu}^{\infty} (k+1)^r e^{-ak^2} \tag{2.3}$$

[1] Olver's analysis is restricted to the case $0 < \alpha \le 4$.