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Fractional variational calculus for nondifferentiable functions

Ricardo Almeida *, Delfim F.M. Torres

Department of Mathematics, University of Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal

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ABSTRACT

We prove necessary optimality conditions, in the class of continuous functions, for variational problems defined with Jumarie's modified Riemann–Liouville derivative. The fractional basic problem of the calculus of variations with free boundary conditions is considered, as well as problems with isoperimetric and holonomic constraints.

1. Introduction

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann–Liouville and the Caputo derivatives. Each fractional derivative presents some advantages and disadvantages (see, e.g., [1–3]). The Riemann–Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero but demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense [4].

Recently, Guy Jumarie (see [5–11]) proposed a simple alternative definition to the Riemann–Liouville derivative. His modified Riemann–Liouville derivative has the advantages of both the standard Riemann–Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. Here we show that Jumarie's derivative is more advantageous for a general theory of the calculus of variations.

The fractional calculus of variations is a recent research area much in progress. It is being mainly developed for Riemann–Liouville (see, e.g., [12–17]) and Caputo derivatives (see, e.g., [18–23]). For more on the calculus of variations, in terms of other fractional derivatives, we refer the reader to [24–28] and references therein.

As pointed out in [29], the fractional calculus of variations in Riemann–Liouville sense, as it is known, has some problems, and results should be used with care. Indeed, in order for the Riemann–Liouville derivatives ${}_aD_x^\alpha y(x)$ and ${}_xD_b^\alpha y(x)$ to be continuous on a closed interval $[a, b]$, the boundary conditions $y(a) = 0$ and $y(b) = 0$ must be satisfied [4]. This is very restrictive when working with variational problems of minimizing or maximizing functionals subject to arbitrarily given boundary conditions, as often done in the calculus of variations (see Proposition 1 and Remark 2 of [14]). With Jumarie's fractional derivative this situation does not occur, and one can consider general boundary conditions $y(a) = y_a$ and $y(b) = y_b$.

* Corresponding author.
E-mail addresses: ricardo.almeida@ua.pt (R. Almeida), delfim@ua.pt (D.F.M. Torres).

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The paper is organized as follows. In Section 2 we state the assumptions, notations, and the results of the literature needed in the sequel. Section 3 reviews Jumarie's fractional Euler–Lagrange equations [30]. Our contribution is then given in Section 4: in Section 4.1 we consider the case when no boundary conditions are imposed on the problem, and we prove associated transversality (natural boundary) conditions; optimization with constraints (integral or not) are studied in Sections 4.2 and 4.3. Finally, in Section 5 we explain the novelties of our results with respect to previous results in the literature.

2. Preliminaries on Jumarie's Riemann–Liouville derivative

Throughout the text $f : [0, 1] \to \mathbb{R}$ is a continuous function and α a real number on the interval $(0, 1)$. Jumarie's modified Riemann–Liouville fractional derivative is defined by

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-t)^{-\alpha}(f(t) - f(0))\,\mathrm{d}t.$$

If $f(0) = 0$, then $f^{(\alpha)}$ is equal to the Riemann–Liouville fractional derivative of f of order α. We remark that the fractional derivative of a constant is zero, as desired. Moreover, $f(0) = 0$ is no longer a necessary condition for the fractional derivative of f to be continuous on $[0, 1]$.

The $(\mathrm{d}t)^\alpha$ integral of f is defined as follows:

$$\int_0^x f(t)(\mathrm{d}t)^\alpha = \alpha \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t.$$

For a motivation of this definition, we refer to [5].

Remark 2.1. This type of fractional derivative and integral has found applications in some physical phenomena. The definition of the fractional derivative via difference reads

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha}, \quad 0 < \alpha < 1,$$

and obviously this contributes some questions on the sign of h, as it is emphasized by the fractional Rolle's formula $f(x+h) \cong f(x) + h^\alpha f^{(\alpha)}(x)$. In a first approach, in a realm of physics, when h denotes time, then this feature could picture the irreversibility of time. The fractional derivative is quite suitable to describe dynamics evolving in space which exhibit coarse-grained phenomenon. When the point in this space is not infinitely thin but rather a thickness, then it would be better to replace $\mathrm{d}x$ by $(\mathrm{d}x)^\alpha$, $0 < \alpha < 1$, where α characterizes the grade of the phenomenon. The fractal feature of the space is transported on time, and so both space and time are fractal. Thus, the increment of time of the dynamics of the system is not $\mathrm{d}x$ but $(\mathrm{d}x)^\alpha$. For more on the subject see, e.g., [10,30–33].

Our results make use of the formula of integration by parts for the $(\mathrm{d}x)^\alpha$ integral. This formula follows from the fractional Leibniz rule and the fractional Barrow's formula.

Theorem 2.2 (*Fractional Leibniz Rule [34]*)**.** *If f and g are two continuous functions on* $[0, 1]$, *then*

$$(f(x)g(x))^{(\alpha)} = (f(x))^{(\alpha)} g(x) + f(x)(g(x))^{(\alpha)}. \tag{1}$$

Kolwankar obtained the same formula (1) by using an approach on Cantor space [35].

Theorem 2.3 (*Fractional Barrow's Formula [8]*)**.** *For a continuous function f, we have*

$$\int_0^x f^{(\alpha)}(t)(\mathrm{d}t)^{(\alpha)} = \alpha!(f(x) - f(0)),$$

where $\alpha! = \Gamma(1+\alpha)$.

From Theorems 2.2 and 2.3 we deduce the following formula of integration by parts:

$$\begin{aligned}\int_0^1 u^{(\alpha)}(x)v(x)\,(\mathrm{d}x)^\alpha &= \int_0^1 (u(x)v(x))^{(\alpha)}\,(\mathrm{d}x)^\alpha - \int_0^1 u(x)v^{(\alpha)}(x)\,(\mathrm{d}x)^\alpha \\ &= \alpha![u(x)v(x)]_0^1 - \int_0^1 u(x)v^{(\alpha)}(x)\,(\mathrm{d}x)^\alpha.\end{aligned}$$

It has been proved that the fractional Taylor series holds for nondifferentiable functions. See, for instance, [36]. Another approach is to check that this formula holds for the Mittag-Leffler function, and then to consider functions which can be approximated by the former. The first term of this series is Rolle's fractional formula which has been obtained by Kolwankar and Jumarie and provides the equality $\mathrm{d}^\alpha x(t) = \alpha!\mathrm{d}x(t)$.

It is a simple exercise to verify that the fundamental lemma of the calculus of variations is valid for the $(\mathrm{d}x)^\alpha$ integral (see, e.g., [37] for a standard proof):