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A discrete scheme of Laplace–Beltrami operator and its convergence over quadrilateral meshes☆

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Abstract

Laplace–Beltrami operator and its discretization play a central role in the fields of image processing, computer graphics, computer aided geometric design and so on. In this paper, a discrete scheme for Laplace–Beltrami operator over quadrilateral meshes is constructed based on a bilinear interpolation of the quadrilateral. Convergence results for the proposed discrete scheme are established under some conditions. Numerical results which justify the theoretical analysis are also given.

Keywords: Laplace–Beltrami operator; Mean curvature; Quadrilateral meshes; Discretization; Convergence; Bilinear interpolation

1. Introduction

Let $\mathcal{M} \subset \mathbb{R}^3$ be a given sufficiently smooth surface. Laplace–Beltrami operator (LBO) over $\mathcal{M}$, denoted by $\Delta_{\mathcal{M}}$ in this paper, is a generalization of the classical Laplacian Δ from flat space to $\mathcal{M}$. Laplace–Beltrami operator, which relates closely to the mean curvature normal H of surface $\mathcal{M}$ by the relation $\Delta_{\mathcal{M}} p = 2H(p)$ ($p \in \mathcal{M}$), plays a central role in many areas, such as image processing (see [1–3]), surface processing (see [4,5] for references) and the study of geometric partial differential equations (see [6]). In these application areas, the objective surfaces to be processed are usually represented as discrete meshes. Hence, there are comprehensive needs in practice to discretize the LBO and the mean curvature normal H.

Previous work. It is well-known that the most often used and studied meshes in surface processing are triangular and quadrilateral. For the triangular meshes, which are even more popular than quadrilateral meshes, several discrete schemes of LBO have been proposed and used (see [7–13]). These schemes can be expressed by weighted averages over the neighborhood of mesh vertices. Specifically, let M be a triangulation of surface $\mathcal{M}$ with vertices $\{p_i\}$, f be

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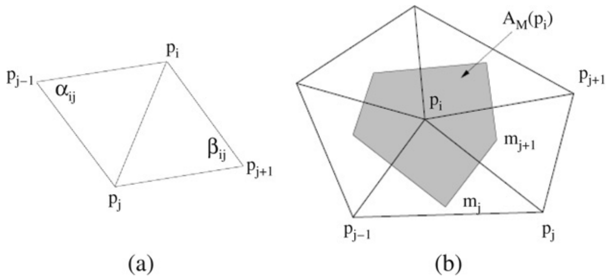

Fig. 1.1. (a) The definition of angles α_{ij} and β_{ij}. (b) The Voronoi region of p_i, where m_j is the circumcenter of triangle $[p_{j-1}p_jp_i]$.

a smooth function defined on $\mathcal{M}$. Then the approximate LBO acting on f is expressed as

$$\Delta_{\mathcal{M}} f(p_i) \approx \sum_{j\in N(i)} w_{ij}(f(p_j) - f(p_i)), \tag{1.1}$$

where $N(i)$ is the index set of one-ring neighbor vertices of p_i, and the weights w_{ij} can be chosen in many ways. For instance, w_{ij} can be taken as $1/\|p_j - p_i\|$ (see [8]), $(\cot\alpha_{ij} + \cot\beta_{ij})/\sum_j(\cot\alpha_{ij} + \cot\beta_{ij})$ (see [7]), $3(\cot\alpha_{ij} + \cot\beta_{ij})/(2A(p_i))$ or $(\cot\alpha_{ij} + \cot\beta_{ij})/(2A_M(p_i))$ (see [9]), and so on. Here α_{ij} and β_{ij} are the two angles opposite to the edge in two triangles sharing the edge $[p_ip_j]$ (see Fig. 1.1(a)), $A(p_i)$ is the summation of areas of triangles surrounding vertex p_i and $A_M(p_i)$ is the area of the Voronoi region of p_i (see Fig. 1.1(b)).

The convergence problem of discrete LBO over triangular meshes has been studied recently in [13,14]. None of the discrete schemes aforementioned have been proved to be convergent over any triangulated surfaces. But some of them converge to the exact LBO under particular conditions. The most important and popular one is the following Desbrun et al.'s discretization:

$$\Delta_{\mathcal{M}} f(p_i) = \frac{3}{2A(p_i)} \sum_j (\cot\alpha_{ij} + \cot\beta_{ij})[f(p_j) - f(p_i)].$$

It converges to the LBO under the conditions that the valence of the vertex p_i is 6 and $p_i = F(q_i)$, $p_j = F(q_j)$ for a smooth parametric surface F and the relations $q_{j+3} + q_j = 2q_i$ $(j = 1, 2, 3)$ hold, where q_j $(j = 1, \ldots, 6)$ are one-ring neighbors of vertex q_i on the 2D domain (see [14] for details).

It is obvious that quadrilateral meshes can be processed as triangular meshes by subdividing each quadrilateral into two triangles. Hence, the discrete schemes of LBO over triangular meshes could be easily applied to quadrilateral ones. However, two ways of subdividing each quadrilateral into triangles often lead to different computational results even though the same discrete scheme is applied to the same quadrilateral. Therefore, it is necessary to construct a discrete scheme which can be used to compute the LBO and the mean curvature normal directly over quadrilateral meshes.

Our work. Discrete schemes of LBO and mean curvature operator are usually derived by minimizing surface mesh area. However, since the vertices of a quadrilateral may not locate on a plane, there is an uncertainty in determining the surface area. The basic idea in the construction of our discrete scheme of LBO in this paper is to use a bilinear interpolation surface of a quadrilateral to represent the polygon. The discrete LBO in the form (1.1) as well as the mean curvature normal is then derived by locally minimizing the area of the interpolation surface. Thus the discrete scheme is uniquely determined. Furthermore, we show that the discrete operators converge to the exact ones in a quadratic rate under some conditions. A preliminary version of this work was reported in a conference [15].

The rest of the paper is organized as follows. In Section 2, we first propose the discretization scheme and then present the convergence results. Simplified discrete schemes are also provided in this section. Since the proofs of these convergence results involve lengthy derivations, we separate them into a single section (Section 3). In Section 4, some numerical experiments are given to show the convergence properties of our discrete scheme. Section 5 includes the conclusion of this paper and future work.