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The exp-function method and generalized solitary solutions

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ABSTRACT

The exp-function method is applied to a system of nonlinear partial differential equations, and generalized solitary solutions with free parameters are obtained. The solution procedure is simple with the help of symbolic computation.

1. Introduction

This paper is concerned with a system of nonlinear partial differential equations (PDEs) given by

$$\begin{aligned} &u_{xt} + a v_x v_t = 0 \\ &v_t + v_{xxx} + (v_x)^3 + b v_x u_{xx} = 0, \end{aligned} \tag{1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are sufficiently often differentiable functions. Many studies about this system have been done in the open literature [1–3]; a physical understanding of the system is also given in [1–3].

There are many results on the exact solutions of nonlinear equations where the initial or boundary conditions are not considered. These solutions are called mathematical solutions because the physical constraints on the real-world problem that is being modeled is not accounted for. Many mathematical solutions for Eq. (1) could be found that carry no physical meaning ($u = v = 1$, for instance, is an exact solution that has no physical meaning at all). Other researchers, on the other hand, begin with some very good initial conditions. Our main aim, however, is to find a generalized solution with some free parameters, which can be identified using the real initial/boundary conditions. To this end, the exp-function method [4,5] will be used.

2. The exp-function method

The exp-function method was first proposed by He and Wu in 2006 [4]. The method attracted immediate attention after its introduction, and it has been now widely applied to search for exact solutions of various physical problems [6–26]. The method is also very effective for stochastic equations [19], and various modifications have appeared, among which the double exp-function method [13] is very effective for the treatment of wave solutions with different velocities and frequencies.

According to the exp-function method, the traveling wave transformation is used:

$$u = u(\eta), \quad \eta = kx + \omega t. \tag{2}$$

Eq. (1) is converted to a system of ordinary differential equations (ODEs):

$$\begin{aligned} &u'' + a\left(v'\right)^2 = 0 \\ &\omega v' + k^3 v''' + k^3 \left(v'\right)^3 + b k^3 v' u'' = 0. \end{aligned} \tag{3}$$

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Substituting $u'' = -a\left(v'\right)^2$ from the first equation of Eq. (3) into the second equation gives

$$\omega v' + k^3 v''' + k^3 (1-ab)\left(v'\right)^3 = 0, \tag{4}$$

where $ab \neq 1$.

Substituting $w = v'$ into Eq. (4), we obtain

$$\omega w + k^3 w'' + k^3 (1-ab)(w)^3 = 0. \tag{5}$$

According to the exp-function method, the solution of Eq. (5) is assumed to have the form

$$v(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d}\exp(-\mathrm{d}\eta)}{a_p \exp(p\eta) + \cdots + b_{-q}\exp(-q\eta)}. \tag{6}$$

By simple calculation, we have

$$w'' = \frac{c_1 \exp[(c+3p)\eta] + \cdots}{c_2 \exp[4p\eta] + \cdots} \tag{7}$$

$$(w)^3 = \frac{c_3 \exp[3c\eta] + \cdots}{c_4 \exp[3p\eta] + \cdots} = \frac{c_3 \exp[(3c+p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots}, \tag{8}$$

where c_i are determined coefficients only for simplicity.

Balancing the highest order of exp-function in Eqs. (7) and (8), we have $c + 3p = 3c + p$; this leads to the result: $p = c$. Similarly, by balancing the lowest order, we can also obtain $q = d$. Generally we can freely choose the values of c and d, but He and Wu [5] proved that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$, and the trial solution, Eq. (6), reduces to

$$w(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{9}$$

Substituting Eq. (9) into Eq. (5), and using some mathematical software, such as Mathematica or MATLAB, we have

$$\frac{1}{A}\left\{C_0 + C_1 \exp[\eta] + C_2 \exp[2\eta] + C_3 \exp[3\eta] + C_4 \exp[4\eta] + C_5 \exp[5\eta] + C_6 \exp[6\eta]\right\} = 0, \tag{10}$$

where the C_i can be written in explicit form. Setting

$$C_0 = 0, \quad C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad C_5 = 0, \quad C_6 = 0, \tag{11}$$

and solving the above system of algebraic equations, we obtain the following solutions.

Case 1:

$$a_1 = 0, \quad a_0 = a_0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_{-1} = \frac{a_0^2 (1-ab)}{8}, \quad k = k, \quad \omega = -k^3. \tag{12}$$

Cases 2 and 3:

$$a_1 = \pm\sqrt{\frac{2}{ab-1}}, \quad a_0 = 0, \quad a_{-1} = a_{-1}, \quad b_0 = 0, \quad b_{-1} = \mp a_{-1}\sqrt{\frac{ab-1}{2}}, \quad k = k, \quad \omega = 2k^3. \tag{13}$$

Cases 4 and 5:

$$\begin{aligned} &a_1 = \pm\frac{1}{\sqrt{2(ab-1)}}, \quad a_0 = \mp\frac{b_0}{\sqrt{2(ab-1)}}, \quad a_{-1} = 0, \quad b_0 = b_0, \quad b_{-1} = 0, \\ &k = k, \quad \omega = \frac{k^3}{2}. \end{aligned} \tag{14}$$

Cases 6 and 7:

$$\begin{aligned} &a_1 = \pm\frac{1}{\sqrt{2(ab-1)}}, \quad a_0 = 0, \quad a_{-1} = \mp\frac{b_0^2}{4\sqrt{2(ab-1)}}, \quad b_0 = b_0, \quad b_{-1} = \frac{b_0^2}{4}, \\ &k = k, \quad \omega = \frac{k^3}{2}. \end{aligned} \tag{15}$$

Cases 8 and 9:

$$\begin{aligned} &a_1 = \pm\frac{1}{\sqrt{2(ab-1)}}, \quad a_0 = a_0, \quad a_{-1} = 0, \quad b_0 = \mp a_0\sqrt{2(ab-1)}, \quad b_{-1} = 0, \\ &k = k, \quad \omega = \frac{k^3}{2}. \end{aligned} \tag{16}$$