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A new method for solving boundary value problems for partial differential equations

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ABSTRACT

This paper proposes a symmetry–iteration hybrid algorithm for solving boundary value problems for partial differential equations. First, the multi-parameter symmetry is used to reduce the problem studied to a simpler initial value problem for ordinary differential equations. Then the variational iteration method is employed to obtain its solution. The results reveal that the proposed method is very effective and can be applied for other nonlinear problems.

1. Introduction

Nonlinear phenomena appear everywhere in our daily life and in engineering applications. Generally, nonlinear equations are very difficult to solve explicitly. In recent years, many power methods have been proposed for finding exact solutions of nonlinear problems, such as the tanh-function method [1] and the homogeneous balance method [2], but up to now, there has not been a single unified method that can solve all kinds of nonlinear equations, among which the boundary value problems are even more different to deal with, and many asymptotic methods have been suggested, such as the homotopy perturbation method [3]. Another asymptotic method is the variational iteration method, which was proposed by Ji-Huan He [4] in 1999 and developed by many researchers, and now it has become a widely used method [5–9], but the variational iteration method is not very effective for solving partial differential equations, though much effort had been made by many authors. For example, Mohyud-Din et al. employed a complex transformation to convert partial differential equations into ordinary differential equations [10]; Hesameddini and Latifizadeh constructed variational iteration algorithms using the Laplace transform [11]. In this paper we will suggest a symmetry–iteration hybrid algorithm.

2. The symmetry–iteration hybrid algorithm

In order to solve PDEs more effectively using the variational iteration method, we will convert PDEs into ODEs using Lie transformations [12,13] so that the variational iteration method can be easily applied. In order to illustrate the basic idea of the symmetry–iteration hybrid algorithm, we consider the boundary layer viscous flow over a stretched impermeable plate, governed by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2},\qquad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0, \tag{1}$$

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subject to the boundary conditions

$$y = 0 : u(x, 0) = \Psi_1(x), \qquad \upsilon(x, 0) = \Psi_2(x),$$
$$y \to \infty : u \to 0, \quad x > 0 \tag{2}$$

where x, y denote the Cartesian coordinates along the plate and normal to it, u, υ are velocity components of fluid in the x, y directions, υ is the kinematic viscosity.

Introducing the stream function ψ defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad \upsilon = \frac{\partial \psi}{\partial x}.$$

Eq. (1) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon \frac{\partial^3 \psi}{\partial y^3}, \tag{3}$$

with the following boundary conditions:

$$\psi_x(x, 0) = \Psi_1(x), \qquad \psi_y(x, 0) = \Psi_2(x), \qquad \psi_y(x, +\infty) = 0. \tag{4}$$

We will look for any possible multiple-parameter symmetries of the above boundary value problem, Eqs. (3) and (4). The infinitesimal generator associated with the symmetries has the general form

$$V = \xi(x, y, \psi)\frac{\partial}{\partial x} + \tau(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}. \tag{5}$$

By the method given in [14–16], we obtain the following infinitesimal functions:

$$\xi = \alpha x + \theta, \qquad \tau = \delta y + \gamma x, \qquad \eta = (\alpha - \delta)\psi + k, \tag{6}$$

where $\alpha, \theta, \delta, \gamma, k$ are arbitrary symmetry parameters. Therefore, Eq. (3) admits five finite-parameter symmetries. These five parameters are adjusters for simplifying the problem. Starting with the invariance of the boundary surface $y = 0$, we can write the boundary surface of the problem under study as follows:

$$\omega_1(x, y) = y, \qquad \omega_2(x, y) = y - K, \quad K \in R^+ \text{ (non-negative real number set)}. \tag{7}$$

Thus, the invariance requirement for the former takes the form

$$V(\omega_1) = 0, \quad \text{when } \omega_1 = 0 \Rightarrow \tau(x, 0) = 0. \tag{8}$$

This results in $\gamma = 0$. Eq. (6) is, then, simplified to

$$\xi = \alpha x + \theta,$$
$$\tau = \delta y,$$
$$\eta = (\alpha - \delta)\psi + k. \tag{9}$$

What remains is to require invariance of boundary value. This requires

$$V^{(1)}(-\psi_x - \Psi_1(x)) = 0 \quad \text{when } -\psi_x(x, 0) = \Psi_1(x),$$
$$V^{(1)}(-\psi_y - \Psi_2(x)) = 0 \quad \text{when } -\psi_y(x, 0) = \Psi_2(x),$$

where $V^{(1)}$ is the first prolongation of infinitesimal generator V on $(x, y, \psi, \psi_x, \psi_y)$-space.

Examining the above conditions, we obtain the following differential equations:

$$(\alpha x + \theta)\Psi_1'(x) + \delta \Psi_1(x) = 0,$$
$$(\alpha x + \theta)\Psi_2'(x) + (\alpha - 2\delta)\Psi_2(x) = 0, \tag{10}$$

which directly give the admissible forms for the functions Ψ_1 and Ψ_2 as follows:

$$\Psi_1(x) = c_1|\alpha x + \theta|^{\frac{-\delta}{\alpha}},$$
$$\Psi_2(x) = c_2|\alpha x + \theta|^{\frac{\alpha - 2\delta}{\alpha}}. \tag{11}$$

Consequently, a set of boundary conditions conforming to the symmetries should be of the form

$$\psi_x(x, 0) = c_1|\alpha x + \theta|^{\frac{-\delta}{\alpha}},$$
$$\psi_y = c_2|\alpha x + \theta|^{\frac{\alpha - 2\delta}{\alpha}},$$
$$\psi_y(x, +\infty) \to 0, \quad x \succ 0, \tag{12}$$

where c_1, c_2 are arbitrary constants.