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Analytic solutions to the boundary layer problem over a stretching wall

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ABSTRACT

Analytic solutions to similarity boundary layer equations are given for boundary layer flows of Newtonian fluid over a stretching wall with power law stretching velocity. The existence of analytic solutions is proven. The Crane's solution is generalized and recurrence relations are obtained for the determination of coefficients of the exponential series.

1. Governing equations for boundary layers

The problem considered here is the steady boundary layer flow due to a moving flat surface in an otherwise quiescent Newtonian fluid medium moving at a speed of $U_w(x)$. In the absence of body force and an external pressure gradient, laminar boundary layer equations expressing conservation of mass and the momentum boundary layer equations for an incompressible fluid are written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where (x, y) are the respective streamwise and plate-normal directions with (u, v) the corresponding velocities, and ν is the kinematic viscosity of the ambient a fluid which will be assumed constant. We consider the boundary-layer flow induced by a continuous surface stretching with velocity $U_w(x)$. The surface is assumed in general to be permeable and a lateral suction/injection with a certain velocity distribution $V_w(x)$ is applied. Accordingly, the boundary conditions are

$$u(x, 0) = U_w(x), \qquad v(x, 0) = V_w(x), \qquad \lim_{y\to\infty} u(x, y) = 0. \tag{3}$$

The streamfunction ψ is formulated by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

Eq. (2) reduces to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^2 \psi}{\partial y}. \tag{4}$$

Assume the velocity of the plate is the form

$$U_w(x) = Ax^{\kappa}, \qquad V_w(x) = Bx^{(\kappa-1)/2}$$

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where A, B and κ are constants, $A > 0$. The case $B < 0$ corresponds to the suction and $B > 0$ to the injection of the fluid. If the wall is impermeable then $B = 0$. Under transformation

$$\psi = \sqrt{\frac{2\nu}{A(\kappa+1)}}Ax^{\frac{\kappa+1}{2}}f(\eta), \qquad \eta = \sqrt{\frac{A(\kappa+1)}{2\nu}}yx^{\frac{\kappa-1}{2}}.$$

Eq. (4) can be written

$$f''' + ff'' - \frac{2\kappa}{\kappa+1}f'^2 = 0, \tag{5}$$

and the boundary conditions (3) become

$$f(0) = f_w, \qquad f'(0) = 1, \qquad \lim_{\eta\to\infty} f'(\eta) = 0, \tag{6}$$

where

$$f_w = -B\left[\nu A\frac{\kappa+1}{2}\right]^{-\frac{1}{2}}.$$

Now, the velocity components are given by

$$u(x, y) = Ax^{\kappa}f'(\eta),$$
$$v(x, y) = -\left(\frac{2\nu A}{\kappa+1}\right)^{1/2} x^{(\kappa-1)/2}\left[\frac{\kappa+1}{2}f(\eta) + \frac{\kappa-1}{2}\eta f'(\eta)\right].$$

We note that the same boundary value problem appears for the steady free convection flow over a vertical semi-infinite flat plate embedded in a fluid saturated porous medium of ambient temperature T_∞, and the temperature of the plate is $T_w = T_\infty + \bar{A}x^{\kappa}$. There is difference in the region of κ between the two physical problem. For flows in a porous medium, there is a physical meaning when $-1/2 < \kappa < +\infty$ (see [1]), and for boundary layer flows over a stretching wall $-\infty < \kappa < -1$, and $-1/2 < \kappa < +\infty$ [2].

Banks [2] has proved if the wall is impermeable then the boundary value problems (5)–(6) does not admit a similarity solution when $-1 < \kappa \le -1/2$. Numerical solutions were given in papers [2,1]. For some special cases of κ problems (5)–(6) are exactly solvable. These particular cases are $\kappa = 1$ and $\kappa = -1/3$. For the impermeable case with $\kappa = 1$ we refer to the exact solution by Crane [3] and for the permeable case [4]. For an impermeable case with $\kappa = -1/3$ the exact solution is in [2] and the exact analytic solution for the permeable case by Magyari and Keller [5].

In this paper our goal is to prove the existence of the exponential series solution to the nonlinear boundary value problems (5)–(6). Both for permeable and impermeable cases we give a method for the determination of the coefficients and parameters. Numerical results are also presented.

2. Exact solutions

The exact solutions for some special values of κ are known. These are $\kappa = 1$ and $\kappa = -1/3$.

2.1. $\kappa = 1$

The solution of the boundary-value problems (5)–(6) for the velocity $U_w(x) = Ax$, $(\kappa = 1)$ of an impermeable surface, $V_w(X) = 0$, has been reported by Crane [3]. Thus, the stream function of Crane's problem has the form

$$\psi = \sqrt{\frac{\nu}{A}}Axf(\eta), \qquad \eta = \sqrt{\frac{A}{\nu}}y,$$

where $f(\eta)$ is the solution of the ordinary differential equation

$$f''' + ff'' - f'^2 = 0,$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad \lim_{\eta\to\infty} f'(\eta) = 0.$$

Crane's well known solution for $f(\eta)$ and for the corresponding velocity field reads

$$f(\eta) = 1 - e^{-\eta}, \tag{7}$$

and the velocity components are

$$u(x, y) = Axe^{-\eta},$$
$$v(x, y) = -(\nu A)^{1/2}(1 - e^{-\eta}).$$

For that solution one gets $f''(0) = -1$.