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Splines with boundary conditions

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Abstract

This paper explores spaces of splines satisfying boundary conditions using the long exact sequence for relative homology. We relate them to the boundary complex of Δ, where Δ is a planar simplicial complex which triangulates a pseudomanifold. In this case, there is a natural relationship between the Hilbert polynomials (which measure the dimension of $C_k^r(\widehat{\Delta})$ for $k \gg 0$) of the spline modules.

Keywords: Spline; Hilbert polynomial; Polynomial approximation; Simplicial complex

1. Introduction

Let Δ be a connected finite simplicial complex which is supported on $|\Delta| \subset \mathbf{R}^2$, so that Δ and all of its links are pseudomanifolds. Henceforth, the phrase 'simplicial complex' means a simplicial complex with these properties (unless specified otherwise). Let $(\Delta)^0$, $(\Delta)_i$, and $(\Delta)_i^0$ denote the sets of interior faces, i-dimensional faces and i-dimensional interior faces, respectively. Also, let $r \geq 0$ be an integer and let $R = \mathbf{R}[x, y, z]$. Define the following spaces of splines:
$C_k^r(\Delta) := \{F : |\Delta| \to \mathbf{R} \mid F$ is continuously differentiable of order r and $F|_\sigma$ is a polynomial in $\mathbf{R}[x, y]$ of degree at most k for all $\sigma \in (\Delta)_2\}$

and
$C^r(\widehat{\Delta}) := \{F : |\widehat{\Delta}| \to \mathbf{R} \mid F$ is continuously differentiable of order r and $F|_{\widehat{\sigma}} \in R$ for all $\sigma \in (\Delta)_2\}$,
where $\widehat{\Delta}$ is the join of Δ (embedded in the plane $z = 1$) with the origin in $\mathbf{R}^3$. Note that $C^r(\widehat{\Delta})$ is a finitely generated graded R-module, whose kth graded piece is exactly $C_k^r(\widehat{\Delta})$.

The main purpose of this paper is to use homological algebra to obtain results involving the Hilbert polynomial of the spline module. In particular, we explore the space of splines satisfying conditions on the boundary of the simplicial complex Δ, and show how its Hilbert polynomial compares to the Hilbert polynomial of $C^r(\widehat{\Delta})$.

Billera, in [1], introduced the use of homological algebra in spline theory. Further explorations in this direction were conducted by Billera and Rose, in [2,3], as well as Schenck and Stillman, in [4,5]. For other approaches to the study of spline spaces, see Alfeld and Schumaker, in [6,7], Chui and Wang, in [8], Haas in [9], or Yuzvinsky

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in [10]. In the next section, we recall the homology theory which has been used and some previous results. We also define the Hilbert polynomial and give some of its properties which will be used to obtain the desired conclusion. In Section 3, we obtain a formula by using the long exact sequence obtained from the usual short exact sequence in relative homology.

2. Preliminaries

Throughout this paper, all Δ should be considered as simplicial complexes supported in $\mathbf{R}^2$, though the definitions and examples can be extended to higher dimensions.

Definition 2.1. Let R be a ring. A complex $\mathcal{F}$ of R-modules on $(\Delta)^0$ consists of the following:

(a) For each $\sigma \in (\Delta)^0$, an R-module $\mathcal{F}(\sigma)$ and
(b) for each $i \in \{0, \ldots, d\}$, an R-module homomorphism

$$\partial_i : \bigoplus_{\sigma_i \in (\Delta)_i^0} \mathcal{F}(\sigma_i) \longrightarrow \bigoplus_{\sigma_{i-1} \in (\Delta)_{i-1}^0} \mathcal{F}(\sigma_{i-1})$$

such that $\partial_{i-1} \circ \partial_i = 0$.

Example 2.2. For $R = \mathbf{R}[x, y, z]$ and Δ embedded in $\mathbf{R}^2$, fix an integer $r \geq 0$ and let I_τ be generated by l_τ, the homogenization of the linear form vanishing on $\tau \in (\Delta)_1^0$. Define the complex $\mathcal{J}[\Delta]$, whose maps are induced from the usual relative simplicial boundary operators on $\mathcal{R}[\Delta]$, as follows:

$$\begin{aligned}
&\mathcal{J}(\sigma) = 0 \quad \text{for } \sigma \in (\Delta)_2 \\
&\mathcal{J}(\tau) = I_\tau^{r+1} \quad \text{for } \tau \in (\Delta)_1^0 \\
&\mathcal{J}(v) = \sum_{v \in \tau} I_\tau^{r+1} \quad \text{for } v \in (\Delta)_0^0.
\end{aligned}$$

Example 2.3. Define the complex $\mathcal{R}/\mathcal{J}[\Delta]$ defined on $(\Delta)^0$ as the quotient of the complex $\mathcal{R}[\Delta]$ (which computes relative homology with R coefficients) and the complex $\mathcal{J}[\Delta]$ as defined above. Again the maps are induced from the usual simplicial boundary maps.

Given a complex $\mathcal{F}$ of R-modules on $(\Delta)^0$,

$$0 \longrightarrow \bigoplus_{\sigma \in (\Delta)_2} \mathcal{F}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau \in (\Delta)_1^0} \mathcal{F}(\tau) \xrightarrow{\partial_1} \bigoplus_{v \in (\Delta)_0^0} \mathcal{F}(v) \longrightarrow 0$$

let $H_*(\mathcal{F})$ denote the homology of the complex $\mathcal{F}$. Since Δ is two-dimensional, for a short exact sequence of complexes on $(\Delta)^0$:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

the corresponding long exact sequence in homology is:

$$0 \longrightarrow H_2(\mathcal{A}) \longrightarrow H_2(\mathcal{B}) \longrightarrow H_2(\mathcal{C}) \longrightarrow H_1(\mathcal{A}) \longrightarrow \cdots \longrightarrow H_0(\mathcal{C}) \longrightarrow 0.$$

We recall some results of previous work applying homological algebra to study spline theory. The first lemma relates the second homology module of the quotient complex $\mathcal{R}/\mathcal{J}[\Delta]$ with the spline space of a simplicial complex. The second lemma describes $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ and $H_0(\mathcal{R}/\mathcal{J}[\Delta])$.

Lemma 2.4 (*See [1]*). *Let Δ be a connected finite simplicial complex. If $\mathcal{R}/\mathcal{J}[\Delta]$ is the complex defined in Example 2.3, then $C^r(\widehat{\Delta})$ is isomorphic to the module $H_2(\mathcal{R}/\mathcal{J}[\Delta])$.*

Lemma 2.5 (*See [5]*). *The R-module $H_1(\mathcal{R}/\mathcal{J}[\Delta])$ has finite length, and $H_0(\mathcal{R}/\mathcal{J}[\Delta]) = 0$.*

Now, for a finitely generated graded module M over the polynomial ring $R = \mathbf{R}[x, y, z]$ there is a polynomial (the *Hilbert polynomial*) $\mathrm{HP}(M, t) \in \mathbf{Z}[t]$ such that $\dim_{\mathbf{R}} M_l = \mathrm{HP}(M, l)$ for $l \gg 0$ (See [11], Theorem 1.11). The Hilbert polynomial satisfies some key facts (See [11]).