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Existence for unbounded positive solutions of Schrödinger equations in two-dimensional exterior domains☆

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Abstract

In this paper, by using the fixed point theory, under quite general conditions on the nonlinear term we obtain a existence result of unbounded positive solutions of Schrödinger equations in two-dimensional exterior domains.

Keywords: Positive solution; Schrödinger equations; Fixed point theorem

1. Introduction

In this paper, we consider the existence of unbounded positive solutions of the semilinear Schrödinger equation

$$\triangle u + f(x, u(x)) = 0, \quad x \in U_k \tag{1}$$

where $k > 0$, $U_k = \{x \in \mathbf{R}^2 : |x| > k\}$ and f is locally Hölder continuous in $U_k \times \mathbf{R}$.

In recent years, by using the Perron method, A. Constantin investigated the existence of unbounded positive solutions of (1) and obtained many good results. (See [6,10].) But in these results, the conditions which $f(x, 0) \geq 0$, and for some $c > 0$ and some $c > \epsilon > 0$ there exists a dominating function $h(|x|, u(x))$ with $h(|x|, u(x)) \geq |f(x, u(x))|$ such that for any $p(t) \in C([0, \infty))$ with $|p(t) - c| \leq \epsilon$ for $t \geq 0$,

$$\int_0^\infty \mathrm{e}^{2m} h(\mathrm{e}^m, mp(m))\mathrm{d}m \leq \infty \quad \text{for } t \geq 0, \tag{2}$$

were assumed.

When the nonlinearity $f(x, u(x))$ is radial, i.e., $f(x, u(x)) = f(|x|, u(x))$, a positive solution of (1) comes right from the ODE

$$p''(s) + \mathrm{e}^{2s} f(\mathrm{e}^s, p(s)) = 0, \quad s \geq 0. \tag{3}$$

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For the existence of unbounded positive global solutions of (3), the reader is referred to the papers [2–11]. But in these results, the conditions that for some $c > 0$, some $\epsilon > 0$ and any $p(t) \in C([0, \infty))$ with $|p(t) - c| \leq \epsilon$ for $t \geq 0$,

$$\int_t^\infty e^{2m} f(e^m, mp(m)) dm \quad \text{for } t \geq 0 \quad \text{or}$$

$$\int_t^\infty e^{2m} \left| f(e^m, mp(m)) \right| dm \quad \text{for } t \geq 0 \text{ exists} \tag{4}$$

were assumed.

For comparison with previous works, we see the following two examples.

Example 1.1. Consider the following equation

$$\triangle u + \frac{u \sin \ln |x|}{A|x|^2 (1 + u \ln |x|)^\alpha} = 0, \quad x \in \mathbf{R}^2, \ |x| \geq 1 \tag{5}$$

where A is a constant with $|A| \geq 100$ and $\alpha = 1$.

Remark 1.2. In this example, if taking $\alpha > 1$, by using the previous results (See [6,10]), it is obvious that for enough large $|A|$, (5) has a unbounded positive solution. But the previous results cannot be applied to the case $\alpha = 1$, although the condition $f(x, 0) \geq 0$ is satisfied. In fact, if we take $p(t) = c + \frac{\epsilon}{2} \sin t$, then

$$\int_t^\infty e^{2m} h(e^m, p(m)) dm, \qquad \int_t^\infty e^{2m} f(e^m, p(m)) dm \quad \text{and} \quad \int_t^\infty e^{2m} |f(e^m, p(m))| dm$$

don't exist.

Example 1.3. Consider the following equation

$$\triangle u + \frac{u \sin \ln |x|}{A|x|^2 (1 + u \ln |x|)^\alpha} + \frac{\xi \cdot x}{A|x|^2 (1 + |\xi||x|)(1 + \ln |x|)^3} = 0, \quad x \in \mathbf{R}^2, \ |x| \geq 1 \tag{6}$$

where $\alpha > 1$, A is a constant with $|A| \geq 100$ and $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ where ξ_1 and ξ_1 are constants with $\xi_1 \times \xi_2 \neq 0$.

Remark 1.4. In this example, although the conditions: (2) and (4) are satisfied, the previous results cannot be applied to the case where $f(x, 0)$ is sign-changing.

Therefore, it is a significant work to investigate the existence of unbounded positive solutions of (1) under the condition where $f(x, 0)$ is sign-changing, or the condition where $\int_t^\infty e^{2m} h(e^m, mp(m)) dm$ exists is relaxed (or where $\int_t^\infty e^{2m} f(e^m, mp(m)) dm$ exists is relaxed.). Hence, it is much more significant work to investigate the existence of unbounded positive solutions of (1) under the conditions where $f(x, 0)$ is sign-changing and $\int_t^\infty e^{2m} h(e^m, mp(m)) dm$ and $\int_t^\infty e^{2m} f(e^m, mp(m)) dm$ exist are relaxed. The latter is our aim in this paper. (See Example 3.3.)

In Section 3, we shall prove that (5) and (6) have unbounded positive solutions and give a new example.

In Section 2, we consider the existence of unbounded positive global solutions of the equation

$$u''(t) + g(t, u(t)) = 0, \quad t \geq 0 \tag{7}$$

where $g \in C([0, \infty) \times \mathbf{R}, \mathbf{R})$.

For the existence of unbounded positive global solutions of (7), the reader is referred to the papers [2–11]. But in these results, the condition that for some $c > 0$, some $\epsilon > 0$ and any $u(t) \in C([0, \infty))$ with $|u(t) - c| \leq \epsilon$ for $t \geq 0$,

$$\int_t^\infty g(m, mu(m)) dm \quad \text{for } t \geq 0 \quad \text{or} \quad \int_0^\infty |g(m, mu(m))| dm \text{ exists} \tag{8}$$

or for some $c > 0$ and some $\epsilon > 0$ there exists a dominating function $h(t, u(t))$ with $h(t, u(t)) \geq |g(t, u(t))|$ for $t \geq 0$ such that for any $u(t) \in C([0, \infty))$ with $|u(t) - c| \leq \epsilon$ for $t \geq 0$,

$$\int_0^\infty h(m, mu(m)) dm \quad \text{exists} \tag{9}$$