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Coincidence and common fixed and periodic point theorems in cone metric spaces

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ABSTRACT

The aim of this paper is to show the existence of coincidence and fixed points for mappings satisfying property (E.A) in cone metric spaces. Also, we give periodic point theorems in cone metric spaces.

1. Introduction and preliminaries

We know that in the setting of metric space, the strict contractive condition do not ensure the existence of common fixed point unless the space is assumed compact or the strict conditions are replaced by strong conditions (see [1–3]). Jungck [4] introduced the concept of compatible mappings. This notion was used to prove the existence of common fixed points. And also, the study of common fixed points of noncompatible mappings is very interesting. Pant [5,6] initially investigated common fixed points of noncompatible mappings defined on metric spaces. Aamri and El Moutawakil [7] defined a property (E.A) which generalizes the concept of noncompatible mappings and gave some common fixed point theorems under strict contractive conditions.

Recently, Huang and Zhang [8] introduced the notion of cone metric spaces as a generalization of metric spaces. They introduced the concept of convergence in cone metric spaces and obtained some fixed point theorems for contractive maps in cone metric spaces. Since then, many authors [8–14] established fixed point theorems in cone metric spaces.

In this paper, we give some coincidence and common fixed point theorems under strict contractive conditions for mappings satisfying property (E.A) in cone metric spaces. These results are generalizations of results in [7,11].

Consistent with Huang and Zhang [8], the following definitions will be needed in what follows.

Let E be a real Banach space. A subset P of E is a *cone* if the following conditions are satisfied:

(i) P is nonempty closed and $P \neq \{0\}$,
(ii) $ax + by \in P$, whenever $x, y \in P$ and $a, b \in \mathbb{R}(a, b \geq 0)$,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

For $x, y \in P$, $x \ll y$ stand for $y - x \in int(P)$, where $int(P)$ is the interior of P. A cone P is called *normal* if there exists a number $K > 0$ such that for all $x, y \in E$, $\|x\| \leq K\|y\|$ whenever $0 \leq x \leq y$.

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A cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{u_n\}$ is a sequence such that for some $z \in E$

$$u_1 \leq u_2 \leq \cdots \leq z,$$

then there exists $u \in E$ such that

$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

It is well known [8] that every regular cone is normal (see also [14]).

From now on, we assume that E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

For a nonempty set X, a mapping $d : X \times X \to E$ is called *cone metric* on X if the following conditions are satisfied:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 1.1. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|,\ \alpha|x - y|)$, where α is a positive constant. Then (X, d) is a cone metric space and P is a regular cone (see [8]).

The following example is a cone metric space with non-normal cone.

Example 1.2. Let $E = C^1_{\mathbb{R}}([0, 1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and $P = \{f \in E : f \geq 0\}$. Then P is a non-normal cone (see [14]).

Let $X = [0, \infty)$. Define $d : X \times X \to E$ by $d(x, y) = |x - y|f$, where $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = e^t$. Then (X, d) is a cone metric space.

A sequence $\{x_n\}$ in a cone metric space (X, d) *converges* to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $c \in int(P)$, there exists N such that for all $n > N$, $d(x_n, x) \ll c$. A sequence $\{x_n\}$ in a cone metric space (X, d) is *Cauchy* if for any $c \in int(P)$, there exists N such that for all $n, m > N$, $d(x_n, x_m) \ll c$. A cone metric space (X, d) is called *complete* if every Cauchy sequence is convergent.

Note that if $\lim_{n\to\infty} d(x_n, x) = 0$, then $\lim_{n\to\infty} x_n = x$. The converse is true if P is a normal cone. Also, if P is a normal cone, then $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Lemma 1.1. *Let P be a cone in Banach space E with $int(P) \neq \emptyset$. Then the following are satisfied.*

(1) *If $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = 0$.*
(2) *If $0 \leq u \ll c$ for each $c \in int(P)$, then $u = 0$.*
(3) *If $0 \leq u_n \leq v_n$ and $u_n \to u$, $v_n \to v$, then $u \leq v$.*

Let S and T be two self-mappings of a cone metric space (X, d).

(1) The pair (S, T) is said to be *compatible* if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\lim_{n\to\infty} d(Tx_n, x) = \lim_{n\to\infty} d(Sx_n, x) = 0$ for some $x \in X$.
(2) The pair (S, T) is said to be *weakly compatible* if $STu = TSu$ whenever $Su = Tu$ for some $u \in X$.
(3) We say that S and T satisfy the property $(E.A)$ if there exist a sequence $\{x_n\}$ in X and a point x in X such that $\lim_{n\to\infty} d(Sx_n, x) = \lim_{n\to\infty} d(Tx_n, x) = 0$.

It is easy to see that two compatible maps are weakly compatible, but the converse is not true as shown in the next example.

Example 1.3. Let (X, d) be given as in Example 1.2. Let $S, T : X \to X$ be the mappings defined by

$$Sx = \begin{cases} 2 - x & (0 \leq x < 1), \\ 2 & (x \geq 1) \end{cases} \quad \text{and} \quad Tx = \begin{cases} x & (0 \leq x < 1), \\ 2 & (x \geq 1). \end{cases}$$

Let $\{x_n\}$ be a sequence in X with $x_n = 1 - \frac{1}{n}$. Then we have

$$\lim_{n\to\infty} d(Sx_n, 1) = \lim_{n\to\infty} \left|\frac{1}{n}\right| f = 0$$

and

$$\lim_{n\to\infty} d(Tx_n, 1) = \lim_{n\to\infty} \left|-\frac{1}{n}\right| f = 0.$$

But $\lim_{n\to\infty} d(TSx_n, STx_n) = \lim_{n\to\infty} d(2, 1 + \frac{1}{n}) = \lim_{n\to\infty} |1 - \frac{1}{n}|f \neq 0$. Thus the pair (S, T) is not compatible.