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Symbolization of generating functions; an application of the Mullin–Rota theory of binomial enumeration

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Abstract

We have found that there are more than a dozen classical generating functions that could be suitably symbolized to yield various symbolic sum formulas by employing the Mullin–Rota theory of binomial enumeration. Various special formulas and identities involving well-known number sequences or polynomial sequences are presented as illustrative examples. The convergence of the symbolic summations is discussed.

Keywords: Generating function; Symbolic sum formula; Binomial enumeration; Shift-invariant operator; Delta operator; Bell number; Genocchi number; Euler number; Euler polynomial; Eulerian fraction; Bernoulli number; Bernoulli polynomial

1. Introduction

It is known that the symbolic calculus with operators Δ (differencing), E (operation of displacement), and D (derivative) plays an important role in the Calculus of Finite Differences, which is often employed by statisticians and numerical analysts. Various well-known results can be found in some classical treatises, e.g., those by Jordan [1], Milne-Thomson [2], etc. Since all the symbolic expressions used and operated in the calculus could be formally expressed as power series in Δ (or D or E) over the real or complex number field, it is clear that the theoretical basis of the calculus may be found within the general theory of the formal power series. Worth reading is a sketch of the theory of formal series that has been given briefly in Comtet [3] (see Section 1.12, and Section 3.2–3.5) (cf. Bourbaki [4] Chap. 4-5).

This paper is a sequel to the authors with Torney paper [5], which can be considered as a special case of our results (see Remark 3.2). In this paper we shall show that a variety of formulas and identities containing famous number sequences, namely Bell, Bernoulli, Euler, Fibonacci, Genocchi, and Stirling, could be quickly derived by using a symbolic method with operators Δ, E, and D. The key idea is a suitable application of a certain symbolic

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substitution rule to the generating functions for those number sequences, so that a number of symbolic expressions could be obtained, which then can be used as stepping-stones to yielding particular formulas or identities of interest.

Frequently we shall get formulas or identities involving infinite series expansions. Certainly, any convergence problems, if involved in the results, should be treated separately.

2. A substitution rule and its scope of applications

As usual, we denote by C^∞ the class of real functions, infinitely differentiable in $\mathbb{R} = (-\infty, \infty)$. We will make frequent use of the operators Δ, E, and D which are known to be defined for all $f \in C^\infty$ via the relations

$$\Delta f(t) = f(t+1) - f(t), \qquad Ef(t) = f(t+1), \qquad Df(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t).$$

Consequently they satisfy some simple symbolic relations such as

$$E = 1 + \Delta, \qquad E = e^D, \qquad \Delta = e^D - 1, \qquad D = \log(1+\Delta), \tag{2.1}$$

where the unity 1 serves as an identity operator I such that $If(t) = f(t) = 1f(t)$, and e^D and $\log(1 + D)$ are meaningful in the sense of formal power series expansions, namely

$$e^D = \sum_{k\geq 0} \frac{1}{k!} D^k, \qquad \log(1+\Delta) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \Delta^k$$

so that $e^D f(t) = \sum_{k\geq 0} D^k f(t)/k! = f(t+1) = Ef(t)$ (cf. Jordan [1]).

An operator T which commutes with the shift operator E is called a *shift-invariant operator* (see, for example, [6]), i.e.,

$$TE^\alpha = E^\alpha T,$$

where $E^\alpha f(t) = f(t+\alpha)$ and $E^1 \equiv E$. Clearly, the identity operator 1, the differentiation operator D, and the difference operator Δ are all shift-invariant operators. A shift-invariant operator Q is called a *delta operator* if Qt is a non-zero constant. Obviously, the identity operator, the differentiation operator, the difference operator, and the backward difference, the central difference, Laguerre, and Abel operators (cf. [6]) are all delta operators.

Note that there are two well-known operational formulas involving Stirling numbers of the first and second kinds, $S_1(m, n)$ and $S_2(m, n)$, namely the following

$$D^m f(t) = \sum_{n\geq m} \frac{m!}{n!} S_1(n, m) \Delta^n f(t) \tag{2.2}$$

$$\Delta^m f(t) = \sum_{n\geq m} \frac{m!}{n!} S_2(n, m) D^n f(t). \tag{2.3}$$

These could be derived using Newton interpolation series and Taylor series, respectively (cf. [1] Sections 56 and 67).

Certainly, according to (2.1), it is obvious that (2.2) and (2.3) may be viewed as direct consequences of the substitutions $t \to \Delta$ and $t \to D$ into the following generating functions, respectively

$$(\log(1+t))^m = \sum_{n\geq m} \frac{m!}{n!} S_1(n, m) t^n$$

$$\left(e^t - 1\right)^m = \sum_{n\geq m} \frac{m!}{n!} S_2(n, m) t^n.$$

Note that certain particular identities could be deduced from (2.2) and (2.3) with particular choices of $f(t)$ (cf., for example, [1]).

The above description is an example of the following general substitution rule shown in Mullin and Rota [6] (see also in [7]).