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On a class of multivalent functions defined by an extended multiplier transformations

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ABSTRACT

In the present paper, the authors investigate starlikeness and convexity of a class of multivalent functions defined by an extended multiplier transformation. As a consequence, a number of sufficient conditions for starlikeness and convexity of analytic functions are also obtained.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. We write $A(1) = A$. A function $f \in A(p)$ is said to be p-valent starlike of order $\alpha (0 \le \alpha < p)$ in U if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U,$$

we denote by $S_p^*(\alpha)$, the class of all such functions. A function $f \in A(p)$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in U if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U,$$

let $C_p(\alpha)$ denote the class of all those functions $f \in A(p)$ which are p-valently convex of order α in U. The class $S_p^*(\alpha)$ was introduced by Patil and Thakare [1] and the class $C_p(\alpha)$ was introduced by Owa [2]. Also we note that the classes S_p^* and K_p were introduced by Goodman [3]. Note that $S_1^*(\alpha) = S^*(\alpha)$ and $C_1(\alpha) = C(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order α $(0 \le \alpha < 1)$. We shall use S^* and C to

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denote $S^*(0)$ and $C(0)$, respectively, which are the classes of univalent starlike and univalent convex functions. In [4] Catas extended the multiplier transformation and defined the operator $I_p^n(\lambda,\ell)$ on $f(z)\in A(p)$ by the following infinite series:

$$I_p^n(\lambda,\ell)f(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^n a_k z^k \quad (\ell\geq 0;\ \lambda\geq 0;\ p\in N \text{ and } n\in N_0=N\cup\{0\}). \tag{1.2}$$

We note that:

$$I_p^0(1,0)f(z)=f(z),\ I_p^1(1,0)f(z)=\frac{zf'(z)}{p} \quad\text{and}\quad I_p^2(1,0)f(z)=\frac{zf'(z)+z^2f''(z)}{p^2}.$$

By specializing the parameters λ, ℓ and p, we obtain the following operators studied by various authors:

(i) $I_p^n(1,\ell)f(z)=I_p(n,\ell)f(z)$ (see [5,6]);
(ii) $I_p^n(1,0)f(z)=D_p^nf(z)$ (see [7–9]);
(iii) $I_1^n(1,\ell)f(z)=I_\ell^nf(z)$ (see [10,11]);
(iv) $I_1^n(1,0)f(z)=D^nf(z)$ (see [12]);
(v) $I_1^n(\lambda,0)f(z)=D_\lambda^nf(z)$ (see [13]);
(vi) $I_1^n(1,1)f(z)=I^nf(z)$ (see [14]);
(vii) $I_p^n(\lambda,0)=D_{\lambda,p}^nf(z)$, where $D_{\lambda,p}^nf(z)$ is defined by

$$D_{\lambda,p}^nf(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{p+\lambda(k-p)}{p}\right]^n a_kz^k. \tag{1.3}$$

Also we note that (see [4]):

$$\lambda z\left(I_p^n(\lambda,\ell)f(z)\right)'=(p+\ell)I_p^{n+1}(\lambda,\ell)f(z)-[p(1-\lambda)+\ell]\,I_p^n(\lambda,\ell)f(z) \quad (\lambda>0) \tag{1.4}$$

and

$$I_p^{n_1}(\lambda,\ell)\left(I_p^{n_2}(\lambda,\ell)f(z)\right)=I_p^{n_2}(\lambda,\ell)\left(I_p^{n_1}(\lambda,\ell)f(z)\right),$$

for all integers n_1 and n_2.

Also if f is given by (1.1), then we have

$$I_p^n(\lambda,\ell)f(z)=\left(\varphi_p^n(\lambda,\ell)*f\right)(z),$$

where

$$\varphi_p^n(\lambda,\ell)(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\lambda(k-p)}{p+\ell}\right]^m z^k.$$

A function $f\in A(p)$ is said to be in the class $S_n(p,\lambda,\ell,\alpha)$ for all z in U if it satisfies

$$\operatorname{Re}\left\{\frac{I_p^{n+1}(\lambda,\ell)f(z)}{I_p^n(\lambda,\ell)f(z)}\right\}>\frac{\alpha}{p} \quad (\ell\geq 0;\ \lambda>0;\ p\in N;\ n\in N_0) \tag{1.5}$$

for some α $(0\leq\alpha<p,\ p\in N)$. We note that $S_0(p,1,0,\alpha)=S_p^*(\alpha)$ and $S_1(p,1,0,\alpha)=C_p(\alpha)$.

In the present paper, our aim is to determine sufficient conditions for a function $f\in A(p)$ to be a member of the class $S_n(p,\lambda,\ell,\alpha)$. As a consequence of our main result, we get a number of sufficient conditions for starlikeness and convexity of analytic functions.

2. Main results

To prove our result, we shall make use of Jack's lemma which we state below.

Lemma 1 ([15]). *Suppose $w(z)$ be a nonconstant analytic function in U with $w(0)=0$. If $|w(z)|$ attains its maximum value at a point $z_0\in U$ on the circle $|z|=r<1$, then $z_0w'(z_0)=\zeta w(z_0)$, where $\zeta\geq 1$ is some real number.*

We now state and prove our main result.

Theorem 1. *If $f\in A(p)$ satisfies*

$$\left|\frac{I_p^{n+1}(\lambda,\ell)f(z)}{I_p^n(\lambda,\ell)f(z)}-1\right|^{\gamma}\left|\frac{I_p^{n+2}(\lambda,\ell)f(z)}{I_p^{n+1}(\lambda,\ell)f(z)}-1\right|^{\beta}<M(p,\lambda,\ell,\alpha,\beta,\gamma) \quad (z\in U), \tag{2.1}$$