Computers and Mathematics with Applications 60 (2010) 653–659

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Fixed points of correspondences defined on cone metric spaces

Zahra Izadi [a], Kourosh Nourouzi [a,b,*]

[a] Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

[b] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

ARTICLE INFO

Article history:
Received 27 August 2009
Received in revised form 13 May 2010
Accepted 14 May 2010

Keywords:
Fixed point
Cone metric space
Correspondence
Banach lattice

ABSTRACT

In the present note, we investigate the fixed points of correspondences defined on cone metric spaces satisfying a conditionally contractive condition.

1. Introduction and preliminaries

The investigation of fixed points of mappings satisfying certain contractive conditions has been a cornerstone of fixed point theory and it has vast area of applications such as nonlinear and adaptive control systems, computing magnetostatic fields, and recurrent networks (see [1–3]).

Recently, a generalization of metric space was given in [4] under the name *cone metric space* in which the range of metric, real numbers, is replaced by an ordered Banach space; in this framework, some results on fixed points of contraction mappings are given. The study of fixed point theorems in cone metric spaces was also carried out by some other authors (see e.g. [5–16]). The aim of this work is to present some results on fixed points of correspondences which are defined on cone metric spaces, and satisfy a generalized contractive condition.

We first recall some concepts and then, give the main results in the following two sections.

A nonempty subset $\mathcal{P}$ of a Banach space E is called a *cone* if:

(i) $\mathcal{P}$ is closed, nonempty, and $\mathcal{P} \neq \{0\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in \mathcal{P}$, imply that $ax + by \in \mathcal{P}$;
(iii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

Every cone $\mathcal{P} \subset E$, induces a partial order $\preceq$ on E defined by $x \preceq y$ whenever $y - x \in \mathcal{P}$. The notation $x \prec y$ indicates that $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} \mathcal{P}$, where $\operatorname{int} \mathcal{P}$ denotes the interior of $\mathcal{P}$. The cone $\mathcal{P}$ is called *normal* if there is a positive real number M such that

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|,$$

for all $x, y \in E$.

In the following we always suppose that E is a Banach space, $\mathcal{P}$ is a cone in E with nonempty interior and $\preceq$ is the partial order induced by $\mathcal{P}$.

* Corresponding author. Fax: +98 21 22853650.
E-mail address: nourouzi@kntu.ac.ir (K. Nourouzi).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.05.013

Definition 1 (*[4]*). Let X be a nonempty set. If the mapping $d : X \times X \to \mathcal{P}$ satisfies the following conditions:

(CM1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(CM3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$,

then d is called a *$\mathcal{P}$-cone metric* on X, and the pair (X, d) is called a *$\mathcal{P}$-cone metric space*. In addition, if X is a vector space, then (X, d) will be called a *$\mathcal{P}$-cone metric vector space.*

Some examples of cone metric spaces can be found in the references list. The following example gives a method to construct cone metric spaces in a simple way.

Example 1. Let $\mathcal{A}$ be a C*-algebra with the positive cone $\mathcal{P}$ (the set of all hermitian elements with non-negative real spectrum). If (X, ρ) is a metric space and $p \in \mathcal{P} \setminus \{0\}$, then the mapping

$$(x, y) \to \rho(x, y)p \quad (x, y \in X),$$

is a $\mathcal{P}$-cone metric on X. More generally, consider the metric spaces (X, ρ_n), where (ρ_n) is bounded from above. If p is a nonzero positive element of the closed unit ball of $\mathcal{A}$ and $\sum_n \alpha_n$ is a convergent series of positive real numbers, then the relation

$$d(x, y) := \sum_{n=1}^{\infty} \alpha_n \rho_n(x, y) p^n \quad (x, y \in X),$$

defines a $\mathcal{P}$-cone metric on X.

According to preceding example, easy cone metrics can be constructed using positive definite matrices in full matrix algebra $\mathbb{M}_n$. Also, every nonnegative real valued function of $C(\Omega)$, where Ω is a compact Hausdorff space gives a cone metric.

Definition 2. Let (X, d) be a $\mathcal{P}$-cone metric space.

- A sequence $\{x_n\}$ of X is called convergent to $x \in X$, if for every $c \gg 0$, there is a positive integer N such that $d(x_n, x) \ll c$, for all $n > N$; then $\{x_n\}$ is said to be *convergent* to x and denoted by $x_n \to x$. If for any $c \gg 0$, $d(x_n, x_m) \ll c$ for all sufficiently large m, n, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent to a point of X, then X is called a *complete $\mathcal{P}$-cone metric space* [4].
- A subset C of X is called *open* if for every $x \in C$ there exists $c \gg 0$ such that $d(x, y) \prec c$ implies $y \in C$.
- A subset $C \subset X$ is said to be *closed* if it contains the limit of all its convergent sequences. And the closure of C denoted by $\overline{C}$ is defined as the set of all points of X that are the limit point of some sequence in C.
- For a subset C of X, the set of all $x \in \overline{C} \cap \overline{C^c}$ is called the *boundary* of C and it is denoted by ∂C.

2. Fixed points of C.C. correspondences

We recall that a correspondence φ from a set X to a set Y assigns to each x in X a (nonempty) subset $\varphi(x)$ of Y. For any subset C of X and correspondence $\varphi : C \twoheadrightarrow X$, an element $x \in C$ is said to be a fixed point if $x \in \varphi(x)$.

Definition 3. Let (X, d) be a complete $\mathcal{P}$-cone metric space and C be a nonempty subset of X. A closed-valued correspondence $\varphi : C \twoheadrightarrow X$ is said to be *conditionally contractive* (briefly, C.C.) if there exists a real constant $k \in [0, 1)$ (C.C. constant), such that for every $x, y \in C$, and $p \in \varphi(x)$, there is $q \in \varphi(y)$ such that $d(p, q) \preceq kd(x, y)$.

Theorem 1. *Every C.C. correspondence $\varphi : X \twoheadrightarrow X$ has a fixed point.*

Proof. Let $\varphi : X \twoheadrightarrow X$ be C.C. with C.C. constant k. Pick any point $x_0 \in X$ and choose $x_1 \in \varphi(x_0)$. There exists $x_2 \in \varphi(x_1)$ such that $d(x_1, x_2) \preceq kd(x_0, x_1)$. By induction, we obtain a sequence $\{x_n\}$ such that for each $n \geq 1$, $x_{n+1} \in \varphi(x_n)$ and

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq kd(x_n, x_{n-1}) \\ &\preceq k^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\preceq k^n d(x_1, x_0). \end{aligned}$$

Therefore, for $m < n$ we have

$$d(x_n, x_m) \preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m);$$