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The scrambling index of primitive digraphs☆

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ABSTRACT

In 2009, Akelbek and Kirkland introduced a useful parameter called the scrambling index of a primitive digraph D, which is the smallest positive integer k such that for every pair of vertices u and v, there is a vertex w such that we can get to w from u and v in D by directed walks of length k. In this paper, we obtain some new upper bounds for the scrambling index of primitive digraphs. Moreover, the maximum index problem, the extremal matrix problem and the index set problem for the scrambling index of various classes of primitive digraphs (e.g. primitive digraphs with d loops, minimally strong digraphs, nearly decomposable digraphs, micro-symmetric digraphs, etc.) are settled, respectively.

1. Introduction

Let $D = (V, E)$ be a digraph with vertex set $V = V(D)$ and arc set $E = E(D)$. Loops are permitted but multiple arcs are not. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), \ldots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. A path is a walk with distinct vertices. A cycle is a closed $u \to u$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, and is denoted by $|W|$. The length of a shortest cycle in D is called the girth of D. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k. The distance from u to v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. Let C_p denote a cycle of length p.

For a digraph D with n vertices, the adjacency matrix of D is defined to be the $n \times n$ matrix $A(D) = (a_{ij})$, where $a_{ij} = 1$ if there is an arc from i to j, and $a_{ij} = 0$ otherwise. Conversely, we can define the associated digraph $D(A)$ of an $n \times n$ Boolean matrix A. For a positive integer l, the lth power of D, denoted by D^l, is the digraph on the same vertex set and with an arc from i to j if and only if $i \xrightarrow{l} j$ in D. It is easy to see that $[D(A)]^l = D(A^l)$. A digraph D is primitive if there exists some positive integer k such that $u \xrightarrow{k} v$ for every pair $u, v \in V(D)$. The smallest such k is called the exponent of D, denoted by $\exp(D)$. Let P_n denote the set of all primitive digraphs of order n [1]. It is well known that D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1 [2].

By using the definition of coefficients of ergodicity, Akelbek and Kirkland [3] provided an attainable upper bound on the second largest modulus of eigenvalues of a primitive matrix that makes use of the so-called scrambling index. The scrambling index of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. It is denoted by $k(D)$. For $u, v \in V(D)$ and $u \neq v$, the local scrambling index of u and v is the number

$$k_{u,v}(D) = \min\left\{k \mid u \xrightarrow{k} w \text{ and } v \xrightarrow{k} w, \text{ for some } w \in V(D)\right\}.$$

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Obviously,

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\} \quad \text{and} \quad k(D) \leq \exp(D).$$

We would like to mention that $k(D)$ and $\exp(D)$ can be also defined by matrices. In fact, $k(D)$ is the smallest positive integer k such that any two rows of $[A(D)]^k$ have at least one positive element in a coincident position, and $\exp(D)$ is the smallest positive integer k such that $[A(D)]^k = J$, where J denote the all 1's matrix. Hence

$$k(A(D)) = k(D) \quad \text{and} \quad \exp(A(D)) = \exp(D).$$

In the study of the exponent, the maximum index problem (MIP), the extremal matrix problem (EMP) and the index set problem (ISP) are the three main problems. For surveys on the exponents of various classes of primitive digraphs, see e.g. [1,2]. Note that the scrambling index gives another characterization of primitivity. It is natural to consider the MIP, EMP and ISP for the scrambling index of various classes of primitive digraphs. Observe that the scrambling index is the competition index [4] in the case of primitive digraphs. In [4], Cho and Kim gave an upper bound on the competition index of a primitive digraph D of order n and girth s. In [3,5], Akelbek and Kirkland improved the upper bound given by Cho and Kim, and then they settled the MIP and EMP for the scrambling index of primitive digraphs of order n. In [6], Kim investigated the competition index of tournaments. In [7], the MIP, EMP and ISP for the scrambling index of primitive symmetric digraphs are studied, respectively. In [8], the authors obtained a bound on the scrambling index of a primitive digraph using Boolean rank.

In Section 2 of this paper, some new upper bounds of $k(D)$ for a primitive digraph D are obtained. In Section 3, we settle the MIP, EMP and ISP for the scrambling index of various classes of primitive digraphs, e.g. primitive digraphs with d loops ($1 \leq d \leq n$), primitive minimally strong digraphs, primitive doubly symmetric digraphs, r-indecomposable digraphs ($r \geq 1$), fully indecomposable digraphs, nearly decomposable digraphs, primitive circulant digraphs, primitive micro-symmetric digraphs, and so on.

2. Bounds on the scrambling index

Let A be the adjacency matrix of a primitive digraph D of order n. Let A^T denote the transposed matrix of A. Let k be an integer, $1 \leq k \leq n$. The k-point exponent of A, denoted by $\exp(A, k)$, is the smallest power of A for which there are k rows with no zero entry [9,1,2]. It is known that $k(A) \leq \exp(A)$. Now we show a relationship between $k(A)$ and $\exp(A^T, 1)$ as follows.

Proposition 2.1. $k(D) = k(A) \leq \exp(A^T, 1) = \exp(D(A^T), 1)$.

Proof. Note that $\exp(A^T, 1)$ is the smallest power of A^T for which there is 1 row with no zero entry. Since $(A^{\exp(A^T,1)})^T = (A^T)^{\exp(A^T,1)}$, there is 1 column with no zero entry for $A^{\exp(A^T,1)}$. By the definition of the scrambling index, the inequality $k(A) \leq \exp(A^T, 1)$ holds. □

For certain classes of primitive digraphs, by the inequality in Proposition 2.1, the MIP for the scrambling index can be easily settled. For example, let S_n denote the set of all primitive symmetric digraphs of order n [2], and let T_n denote the set of all primitive tournaments of order n [4,1]. It is known that $\exp(D, 1) \leq n - 1$ for $D \in S_n$ [2]. Moreover, $A(D) = [A(D)]'$ since $D \in S_n$ is symmetric. Then we have the following result in [7]:

$$k(D) \leq n - 1 \quad \text{for } D \in S_n.$$

Since $\exp(D, 1) \leq 3$ [10,2], and $D([A(D)]') \in T_n$ for $D \in T_n$ ($n > 6$), the result in [4] can be obtained immediately:

$$k(D) \leq 3 \quad \text{for } D \in T_n \ (n > 6).$$

Let D be a primitive digraph. If R is a set of distinct lengths of the elementary cycles in D, then let $d_R(i, j)$ denote the length of the shortest walk from i to j which meets at least one circuit of each length of R [11].

Lemma 2.2 ([11]). *Let $D \in P_n$ and $s_1, \ldots, s_t$ ($s_1 > \cdots > s_t \geq 1$, $t \geq 2$) are relatively prime lengths of circuits of D (that is, $\gcd(s_1, \ldots, s_t) = 1$). Then for $h \geq d_R(i, j) + \phi(s_1, \ldots, s_t)$, there exists a walk of length h from i to j, where $R = \{s_1, \ldots, s_t\}$, and $\phi(s_1, \ldots, s_t)$ is the Frobenius number of $s_1, \ldots, s_t$.*

Theorem 2.3. *Let $D \in P_n$ and $R = \{s_1, \ldots, s_t\}$ ($t \geq 2$) be a set of distinct lengths of cycles $C_{s_1}, \ldots, C_{s_t}$ in D, where $s_1 > \cdots > s_t \geq 1$ and $\gcd(s_1, \ldots, s_t) = 1$. Then*

$$k(D) \leq \sum_{i=1}^{t} (n - s_i) + \phi(s_1, \ldots, s_t) + \left\lfloor \frac{s_t}{2} \right\rfloor.$$