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Another refinement of the Pólya–Szegö inequality

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ABSTRACT

In this paper, the authors make use of certain analytical techniques for nonlinear algebraic equation systems in order to give another refinement of the Pólya–Szegö inequality in a triangle, which is associated with one of Chen's theorems (see Chen (1993) [12] and Chen (2000) [13]). Some remarks and observations, as well as two closely-related open problems, are also presented.

1. Introduction and the main results

For a given triangle ABC, we denote by a, b, c its side-lengths, by S its area, by p its semi-perimeter, and by R and r its circumradius and inradius, respectively.

In the year 1925, Georg Pólya (1887–1985) and Gábor Szegö (1895–1985) ([1, p. 161, Problem 17.1]; see also [2, p. 116]) proved the following beautiful and famous inequality which is known as the *Pólya–Szegö inequality* in the triangle ABC:

$$S \leqq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}, \tag{1.1}$$

which may be compared with Weitzenböck's inequality in the triangle ABC (see, for example, [3, p. 42, Theorem 4.4]; see also [4, p. 112, Section 6.3]):

$$S \leqq \frac{a^2+b^2+c^2}{4\sqrt{3}}$$

as well as another known inequality [3, p. 43, Theorem 4.5]:

$$S \leqq \frac{ab+bc+ca}{4\sqrt{3}}.$$

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From among several extensions and modifications of the Pólya–Szegö inequality (1.1), we first recall the following sharpened version given by Leng [5] (see also [6, p. 194]):

$$S \leqq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}\left(1-\frac{(a-b)^2(b-c)^2(c-a)^2}{(abc)^2}\right)^{\frac{1}{6}}. \tag{1.2}$$

Chen [7] (see also [6,8]), on the other hand, strengthened the Pólya–Szegö's inequality (1.1) as follows:

$$S \leqq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}\left(\frac{2r}{R}\right)^{\frac{1}{3}}. \tag{1.3}$$

More recently, Chen [9] gave a beautifully refined version of the Pólya–Szegö inequality (1.1), which we state here as Theorem 1 below.

Theorem 1. *The best positive constant k for the following inequality:*

$$(abc)^{\frac{2}{3}}-\frac{4}{3}\sqrt{3}S \geqq k\left(\frac{r}{R}\right)\left[(b+c-2a)^2+(c+a-2b)^2+(a+b-2c)^2\right] \tag{1.4}$$

is given by

$$k = F(x_0) \approx 0.12512379476902\cdots,$$

where

$$F(x) := \frac{(x+2)^2}{12x^2(x+1)}\left([4(x+2)^4]^{\frac{1}{3}}-\frac{4}{3}\sqrt{3}\,[(x+1)(x+3)]^{\frac{1}{2}}\right) \quad (x>0)$$

and x_0 is one real root of the following equation:

$$6912(x+1)^3(5x^2+18x+12)^6-(x+2)^8(x+3)^3(x^2-14x-12)^6=0.$$

The main object of this paper is to present yet another refinement of the Pólya–Szegö inequality (1.1) given by Theorem 2 below.

Theorem 2. *The best positive constant k for the following inequality:*

$$\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}-S \geqq kr(R-2r) \tag{1.5}$$

is the real root on the interval $\left(1, \frac{23}{20}\right)$ of the following equation:

$$\begin{aligned}&80621568k^{26}-1169012736k^{24}+2306112768k^{22}-1986308842752k^{20}\\&\quad-27116174063851 2k^{18}-7075252951678008k^{16}-72860319298449837k^{14}\\&\quad-315039331520882532k^{12}+143128010909935188k^{10}+407040335182644176k^{8}\\&\quad+175081049919823564k^{6}-18908198108992k^{4}+539361792k^{2}-5184=0.\end{aligned} \tag{1.6}$$

Furthermore, the constant k has its numerical approximation given by

$$k \approx 1.145209656\cdots.$$

2. Preliminary results and lemmas

In order to prove Theorem 2, we require several lemmas.

Lemma 1. *If the following inequality:*

$$\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}-S \geqq kr(R-2r) \quad (k>0) \tag{2.1}$$

holds true, then

$$0<k\leqq\frac{3}{4}\sqrt{3}.$$

Proof. First of all, Chen [7] (see also [8]) derived the following inequality:

$$\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}} \leqq S\left(\frac{R}{2r}\right)^{\frac{1}{2}}. \tag{2.2}$$