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New types of fuzzy ideals in BCK/BCI-algebras

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ABSTRACT

A generalization of an $(\in, \in \vee q)$-fuzzy ideal of a BCK/BCI-algebra is discussed. Characterizations of an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal are provided. Conditions for an $(\in, \in \vee q_k)$-fuzzy ideal (resp. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal) to be a fuzzy ideal are provided. Using the notion of a fuzzy ideal with thresholds, characterizations of a fuzzy ideal, an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal are discussed.

1. Introduction

The notion of a fuzzy subset was introduced by Zadeh [1] in 1965. In [2], the idea of *fuzzy point* and its *belongingness to* and *quasi-coincidence with* a fuzzy subset were used to define (α, β)-*fuzzy subgroups*, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. This was further studied in detail by Bhakat [3,4], Bhakat and Das [5,6], and Yuan et al. [7]. The concept of $(\in, \in \vee q)$-fuzzy subgroup is a viable generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song [8] discussed general forms of fuzzy interior ideals in semigroups. Also, Jun [9,10] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. A generalization of a fuzzy ideal in a BCK/BCI-algebra is discussed by Jun [11] and Guangji et al. [12]. Ma et al. [13] discussed $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. Zhan and Jun [14] dealt with $(\in, \in \vee q)$-fuzzy BCI-positive implicative (resp., BCI-implicative, BCI-commutative) ideals in BCI-algebras. Zhan et al. [15] considered $(\in, \in \vee q)$-fuzzy p-ideals, $(\in, \in \vee q)$-fuzzy q-ideals and $(\in, \in \vee q)$-fuzzy a-ideals in BCI-algebras.

In this paper, we try to have more general form of an $(\in, \in \vee q)$-fuzzy ideal of a BCK/BCI-algebra. We introduce the notion of an $(\in, \in \vee q_k)$-fuzzy ideal in a BCK/BCI-algebra, and give examples which are $(\in, \in \vee q_k)$-fuzzy ideal but not $(\in, \in \vee q)$-fuzzy ideal. We characterize an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal. We provide conditions for an $(\in, \in \vee q_k)$-fuzzy ideal (resp. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal) to be a fuzzy ideal. Using the notion of a fuzzy ideal with thresholds, we discuss characterizations of a fuzzy ideal, an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal. We finally consider characterizations of a fuzzy ideal, an $(\in, \in \vee q_k)$-fuzzy ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal by using implication operators and the notion of implication-based fuzzy ideals. The important achievement of the study with an $(\in, \in \vee q_k)$-fuzzy ideal is that the notion of an $(\in, \in \vee q)$-fuzzy ideal is a special case of an $(\in, \in \vee q_k)$-fuzzy ideal, and thus so many results in the papers [12,11] are corollaries of our results obtained in this paper.

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2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, \theta)$ of type $(2, 0)$ is called a BCI-*algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = \theta)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = \theta)$,
(III) $(\forall x \in X)\ (x * x = \theta)$,
(IV) $(\forall x, y \in X)\ (x * y = \theta, y * x = \theta \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (\theta * x = \theta)$,

then X is called a BCK-*algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * \theta = x)$,
(a2) $(\forall x, y, z \in X)\ (x * y = \theta \Rightarrow (x * z) * (y * z) = \theta, (z * y) * (z * x) = \theta)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = \theta)$.

A non-empty subset A of a BCK/BCI-algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(d1) $\theta \in A$,
(d2) $(\forall x \in X)\ (\forall y \in A)\ (x * y \in A \Rightarrow x \in A)$.

We refer the reader to the books [16,17] for further information regarding BCK/BCI-algebras.

For any fuzzy subset $\mathcal{A}$ of a set X and any $t \in [0, 1]$ the set

$$U(\mathcal{A}; t) = \{x \in X \mid \mathcal{A}(x) \geq t\}$$

is called a *level subset* of $\mathcal{A}$.

A fuzzy subset $\mathcal{A}$ of a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies:

(d3) $(\forall x \in X)\ (\mathcal{A}(\theta) \geq \mathcal{A}(x))$,
(d4) $(\forall x, y \in X)\ (\mathcal{A}(x) \geq \min\{\mathcal{A}(x * y), \mathcal{A}(y)\})$.

A fuzzy subset $\mathcal{A}$ of a set X of the form

$$\mathcal{A}(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $[x; t]$.

For a fuzzy subset $\mathcal{A}$ of a set X, we say that a fuzzy point $[x; t]$ is

(d5) *contained* in $\mathcal{A}$, denoted by $[x; t] \in \mathcal{A}$, [2] if $\mathcal{A}(x) \geq t$.
(d6) *quasi-coincident* with $\mathcal{A}$, denoted by $[x; t]\mathrm{q}\mathcal{A}$, [2] if $\mathcal{A}(x) + t > 1$.

For a fuzzy point $[x; t]$ and a fuzzy subset $\mathcal{A}$ of a set X, we say that

(d7) $[x; t] \in \vee\mathrm{q}\mathcal{A}$ if $[x; t] \in \mathcal{A}$ or $[x; t]\mathrm{q}\mathcal{A}$.
(d8) $[x; t]\overline{\alpha}\mathcal{A}$ if $[x; t]\alpha\mathcal{A}$ does not hold for $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}\}$.

3. New types of fuzzy ideals

Let k denote an arbitrary element of $[0, 1)$ unless otherwise specified. For a fuzzy point $[x; t]$ and a fuzzy subset $\mathcal{A}$ of X, we say that

(d9) $[x; t]\mathrm{q}_k\mathcal{A}$ if $\mathcal{A}(x) + t + k > 1$.
(d10) $[x; t] \in \vee\mathrm{q}_k\mathcal{A}$ if $[x; t] \in \mathcal{A}$ or $[x; t]\mathrm{q}_k\mathcal{A}$.
(d11) $[x; t]\overline{\alpha}\mathcal{A}$ if $[x; t]\alpha\mathcal{A}$ does not hold for $\alpha \in \{\mathrm{q}_k, \in\vee\mathrm{q}_k\}$.

The following theorem is a generalization of [11, Theorem 3.2].

Theorem 3.1. *Let $\mathcal{A}$ be a fuzzy subset of a BCK/BCI-algebra X. Then the following are equivalent:*

(1) $\left(\forall t \in \left(\frac{1-k}{2}, 1\right]\right)\ (U(\mathcal{A}; t) \neq \emptyset \Rightarrow U(\mathcal{A}; t) \lhd X)$.
(2) *$\mathcal{A}$ satisfies the following assertions:*
 (2.1) $(\forall x \in X)\ \left(\mathcal{A}(x) \leq \max\left\{\mathcal{A}(\theta), \frac{1-k}{2}\right\}\right)$.
 (2.2) $(\forall x, y \in X)\ \left(\min\{\mathcal{A}(x * y), \mathcal{A}(y)\} \leq \max\left\{\mathcal{A}(x), \frac{1-k}{2}\right\}\right)$.