Computers and Mathematics with Applications 60 (2010) 792–802

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Well-posedness of parabolic differential and difference equations

Allaberen Ashyralyev [a,b,*], Yaşar Sözen [a]

[a] Department of Mathematics, Fatih University, 34500, Büyükçekmece, Istanbul, Turkey

[b] Department of Mathematics, ITTU, 74012 Ashgabat, Turkmenistan

ARTICLE INFO

Article history:
Received 25 November 2009
Received in revised form 19 May 2010
Accepted 19 May 2010

Keywords:
Abstract parabolic equation
Banach spaces
Fractional spaces
Well-posedness
Difference scheme

ABSTRACT

We consider the abstract parabolic differential equation $u'(t) + Au(t) = f(t)$, $-\infty < t < \infty$ in a Banach space E with $-A$ the infinitesimal generator of an analytic, exponentially decreasing semigroup $\exp\{-tA\}$ $(t \geq 0)$. The main purpose of this paper is to establish the well-posedness of this equation in $C^{\beta}(\mathbb{R}, E_{\alpha})$, $(\alpha, \beta \in [0, 1])$, and the well-posedness of the corresponding Rothe difference scheme in $\mathcal{C}^{\beta}(\mathbb{R}_{\tau}, E_{\alpha})$, $(\alpha, \beta \in [0, 1])$. Moreover, we apply our theoretical results to obtain new coercivity inequalities for the solution of parabolic difference equations.

1. Introduction

Several types of parabolic equations on the whole real line and parabolic equations on the whole real line with infinite delays or fading memory have been studied in [1, page 19], [2,3], [4, page 398], [5–7] (see also the references therein).

As a physical application of such type of equations, we may refer to [8], where there is an application of integro-differential equations arising in population dynamics. Moreover, parabolic equations on the whole real line are used to describe non-classical models of optics (see [9] and the references therein). Analogous situation may appear in some physical problems when considering effects of extremely prolonged (chaotic or periodic) external actions on a body. In such a case the subject "forgets" its initial conditions and they can be considered as posed at minus infinity. The exact form of the condition is not important and the uniqueness of the solution is determined by other requirements for the behavior of the solution.

In [2], we investigated the well-posedness of the parabolic equation

$$u'(t) + Au(t) = f(t), \quad -\infty < t < \infty \tag{1.1}$$

in $C^{\beta}(\mathbb{R}, E)$, where E is a Banach space with $-A$ the infinitesimal generator of an analytic, exponentially decreasing semigroup. We proved that problem (1.1) is well-posed in the Hölder space $C^{\beta}(\mathbb{R}, E)$, $0 < \beta < 1$, and established the well-posedness of the Rothe difference scheme for (1.1) in $\mathcal{C}^{\beta}(\mathbb{R}_{\tau}, E)$, $0 < \beta < 1$. It is known that from that the well-posedness of (1.1) in $C^{\beta}(\mathbb{R}, E)$ for $\beta = 0$, $\beta = 1$, and the well-posedness of the Rothe difference scheme in $\mathcal{C}^{\beta}(\mathbb{R}_{\tau}, E)$ for $\beta = 0$, $\beta = 1$ do not follow.

In this paper, we extend our previous results [2] to fractional spaces. The well-posedness theorems can also be proved for $\beta = 0$ and $\beta = 1$. Furthermore, we apply our theoretical results to obtain new coercivity inequalities for solutions of parabolic difference equations.

* Corresponding author at: Department of Mathematics, Fatih University, 34500, Büyükçekmece, Istanbul, Turkey. Tel.: +90 212 8663300x2086; fax: +90 212 8663402.

E-mail addresses: aashyr@fatih.edu.tr, aashyr@yahoo.com (A. Ashyralyev).

0898-1221/$ – see front matter Crown Copyright © 2010 Published by Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.05.026

The coercivity inequalities (maximal regularity, well-posedness) are one of the most powerful and popular tools in the study of boundary value problems for parabolic and elliptic differential equations [10–12]. The maximal regularity approach enables one to investigate the general boundary value problems for both elliptic and parabolic differential equations. This approach has been used by many researchers to investigate the well-posedness of local and nonlocal boundary value problems for abstract differential and difference equations in Banach spaces (see [13–16,1,17–20,2,21–30,11], and the references therein).

2. The differential equation

In this section, we consider the abstract parabolic differential equation

$$u'(t) + Au(t) = f(t), \quad -\infty < t < \infty, \tag{2.1}$$

where E is an arbitrary Banach space. Here $u(t)$ and $f(t)$ are, respectively, unknown and given abstract functions defined on the set $\mathbb{R}$ of real numbers with values in E, A is a linear unbounded closed operator acting in E with dense domain $D(A) \subset E$.

We say that $u(t)$ is a *solution* of problem (2.1), if the following are fulfilled:

1. $u(t)$ is continuously differentiable and bounded, its derivative is bounded.
2. The element $u(t)$ belongs to $D(A)$ for all $t \in \mathbb{R}$ and the function $Au(t)$ is continuous and bounded in $\mathbb{R}$.
3. $u(t)$ satisfies equation (2.1).

We refer to a solution of problem (2.1) defined in the above sense as a solution of problem (2.1) in the space $C(E) = C(\mathbb{R}, E)$ of all continuously bounded functions $\varphi(t)$ defined on $\mathbb{R}$ with values in E equipped with the norm

$$\|\varphi\|_{C(E)} = \sup_{t\in\mathbb{R}} \|\varphi(t)\|_E.$$

Problem (2.1) is *well-posed* in $C(E)$ if the following are conditions are satisfied:

1. For each $f(t) \in C(E)$, problem (2.1) is uniquely solvable. It means that an additive and homogeneous operator $u(t) \equiv u(t; f(t))$ acting from $C(E)$ to $C(E)$ is defined and gives the solution of problem (2.1) in $C(E)$. Furthermore, the operators $\frac{d}{dt}(u(t; f(t)))$ and $Au(t; f(t))$ acting in $C(E)$ have these properties.
2. Regarded as an operator from $C(E)$ to $C(E)$, $u(t; f(t))$ is continuous. Namely, inequality
$$\|u(t; f(t))\|_{C(E)} \le M\|f\|_{C(E)}, \tag{2.2}$$
holds for some $1 \le M < \infty$, which is independent of $f(t) \in C(E)$.

It follows from the well-posedness of problem (2.1) in $C(E)$ that the operator $u(t; f(t))$ is continuous in $C(E)$, and the operator $Au(t; f(t))$ is defined on the whole space $C(E)$. The operator A, which acts in the Banach space E with domain $D(A)$, generates via the formula $\mathcal{A}u = Au(t)$ an operator $\mathcal{A}$, which acts in the Banach space $C(E)$ and is defined on the functions $u(t) \in C(E)$ with the property that $Au(t) \in C(E)$. By the fact that the operator A^{-1} exists and is bounded, the operator $\mathcal{A}^{-1}$ exists and is bounded, and hence $\mathcal{A}$ is closed in $C(E)$. Hence, the operator $Au(t; f(t)) = \mathcal{A}(\cdot, f)$ is closed in $C(E)$. It follows from Banach's theorem that this operator is continuous, i.e. for every $f(t) \in C(E)$ the inequality

$$\|Au(t; f(t))\|_{C(E)} \le M\|f\|_{C(E)}, \tag{2.3}$$

is valid, where M is independent of $f(t)$.

Thus, from estimates (2.2) and (2.3) it follows that the *coercivity inequality*

$$\|u'\|_{C(E)} + \|Au(t)\|_{C(E)} \le M\|f\|_{C(E)}$$

is obtained for the solutions of well-posed in $C(E)$ problem (2.1) with some $1 \le M < \infty$, which is independent of $f \in C(E)$ [1].

Throughout the paper, M indicates positive constants which can be different from time to time and we are not interested to make precise. We shall write $M(\alpha, \beta, \ldots)$ to stress the fact that the constant depends only on $\alpha, \beta, \ldots$.

We shall assume that the operator $-A$ generates a semigroup e^{-tA} $(t \ge 0)$ with exponentially decaying norm as $t \to \infty$, i.e. there exist $M \ge 1$, $\delta > 0$ such that

$$\|e^{-tA}\|_{E\to E} \le Me^{-\delta t}. \tag{2.4}$$

Let $v(t)$ be the function defined by

$$\begin{cases} (2A)^{-1}e^{tA}v, & \text{if } t < 0, \\ (2A)^{-1}e^{-tA}v + te^{-tA}v, & \text{if } t \ge 0. \end{cases}$$

Let $v \in D(A)$. Then, $v(t)$ is the solution $C(E)$ of (2.1) with $f(t) = e^{-|t|A}v$.