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On a conjecture about the kth lower multiexponent ☆

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ABSTRACT

In this paper, the conjecture about the kth lower multiexponent $f(n, k)$ proposed by R.A. Brualdi and B. Liu is proved to be true for the following cases: (1) $k = n - i$, where $i = 2, 3, 4, 5$; (2) small n, where $n \leq 8$; (3) the class of primitive micro-symmetric digraphs of order n.

1. Introduction

A digraph D is called primitive if and only if D is strongly connected and the greatest common divisor $g.c.d.(r_1, \ldots, r_s) = 1$, where $\{r_1, \ldots, r_s\}$ is the set of distinct lengths of the directed cycles in D [1]. Let D be a primitive digraph with vertex set $V = \{1, \ldots, n\}$, and let $X \subseteq V$. The exponent of the set X is the least integer m such that for each vertex i of D there exists a walk from at least one vertex in X to i of length m, denoted by $\exp_D(X)$ [2].

In 1990, Brualdi and Liu [2] introduced the kth lower multiexponent of a primitive digraph D with n vertices as follows: for $1 \leq k \leq n$,

$$f(D, k) := \min\{\exp_D(X) \mid X \subseteq V \text{ and } |X| = k\},$$

and

$$f(n, k) := \max_D\{f(D, k)\},$$

where the maximum is taken over all primitive digraphs of order n.

In [2], the authors proved that

$$f(n, k) = \begin{cases} n^2 - 3n + 3, & k = 1, \\ 1, & k = n - 1, \\ 0, & k = n, \end{cases}$$

and they proposed the following conjecture about $f(n, k)$.

Conjecture 1.1 (*[2]*). *For any integers n, k with $2 \leq k \leq n - 2$,*

$$f(n, k) = 1 + (2n - k - 2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

It can be seen that the above equality is also true for $k = 1, n - 1$.

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The kth lower multiexponent of a primitive digraph has been studied by many. In particular, in [3–7], Conjecture 1.1 has been verified for several classes of digraphs, including primitive digraphs with a directed cycle whose length is divisible k, primitive simple graphs, primitive tournaments, primitive symmetric digraphs, etc.

In this paper, we prove that Conjecture 1.1 holds for the following cases:

(1) $k = n-2, n-3, n-4, n-5$;
(2) small n, where $n \leq 8$;
(3) the class of primitive micro-symmetric digraphs of order n.

2. Preliminaries

Let D_n be a primitive digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to n-1$, where $n \geq 2$. It is well known that D_n is called the Wielandt digraph [1]. The kth lower multiexponent of the Wielandt digraph had been investigated in [2,1].

Lemma 2.1 ([2,1]). *Let n, k be positive integers with $1 \leq k \leq n-1$. Then*

$$f(D_n, k) = 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

By Lemma 2.1, D_n is an extremal digraph reaching the bound given by $f(n, k)$.

Let $\overline{D_n}$ be a primitive digraph obtained from D_n by adding an arc $2 \to n$, where $n \geq 4$. Note that D_n is a subdigraph of $\overline{D_n}$, then for $1 \leq k \leq n-1$,

$$f\left(\overline{D_n}, k\right) \leq f(D_n, k).$$

For convenience, by an l-dicycle we mean a directed cycle of length l. The following lemmas give some upper bounds of $f(D, k)$ for different cases, which are useful in this paper.

Lemma 2.2 ([2]). *Let D be a primitive digraph of order n. Suppose that D has a s-dicycle. Then for any integer k with $s \leq k \leq n$,*

$$f(D, k) \leq n-k.$$

Lemma 2.3 ([3]). *Let D be a primitive digraph with n vertices. If there is a s-dicycle intersecting with a $(s+1)$-dicycle (or with a $(s+2)$-dicycle, where s is odd) in D, then for $k < s$, we have*

$$f(D, k) \leq 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

Lemma 2.4 ([1,3]). *Let D be a primitive digraph with n vertices which contains a s-dicycle, where $1 \leq s \leq n-1$. Then for $k|s$,*

$$f(D, k) \leq 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

Let $X \subseteq V(D)$ and let $R_t(X)$ be the set of vertices in D, which can be reached by a walk of length t from some vertex in X, where t is a nonnegative integer.

Lemma 2.5. *Let $H_n^{(1)}$ be a primitive digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to n-3$. Then*

$$f\left(H_n^{(1)}, n-4\right) \leq 7 \quad \text{for } n \geq 7$$

and

$$f\left(H_n^{(1)}, n-5\right) \leq 8 \quad \text{for } n \geq 8.$$

Proof. Let $X_1 = \{1, 2, \ldots, n\} - \{2, 4, n-2, n\}$ be a set of $(n-4)$ vertices, where $n \geq 7$. It is not difficult to verify that

$$R_0(X_1) = X_1, \qquad R_1(X_1) = \{1, 2, \ldots, n\} - \{1, 3, n-1\},$$
$$R_2(X_1) = \{1, 2, \ldots, n\} - \{2, n-2, n\}, \ldots,$$
$$R_6(X_1) = \{1, 2, \ldots, n\} - \{n-2\}, \qquad R_7(X_1) = \{1, 2, \ldots, n\}.$$