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Invariant models in the inversion of gravity and magnetic fields and their derivatives

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ABSTRACT

In potential field inversion problems we usually solve underdetermined systems and realistic solutions may be obtained by introducing a depth-weighting function in the objective function. The choice of the exponent of such power-law is crucial. It was suggested to determine it from the field-decay due to a single source-block; alternatively it has been defined as the structural index of the investigated source distribution. In both cases, when *k*-order derivatives of the potential field are considered, the depth-weighting exponent has to be increased by *k* with respect that of the potential field itself, in order to obtain consistent source model distributions. We show instead that invariant and realistic source-distribution models are obtained using the same depth-weighting exponent for the magnetic field and for its *k*-order derivatives. A similar behavior also occurs in the gravity case. In practice we found that the depth weighting-exponent is invariant for a given source-model and equal to that of the corresponding magnetic field, in the magnetic case, and of the 1st derivative of the gravity field, in the gravity case. In the case of the regularized inverse problem, with depth-weighting and general constraints, the mathematical demonstration of such invariance is difficult, because of its non-linearity, and of its variable form, due to the different constraints used. However, tests performed on a variety of synthetic cases seem to confirm the invariance of the depth-weighting exponent.

A final consideration regards the role of the regularization parameter; we show that the regularization can severely affect the depth to the source because the estimated depth tends to increase proportionally with the size of the regularization parameter. Hence, some care is needed in handling the combined effect of the regularization parameter and depth weighting.

1. Introduction

Most geophysical inversion problems are characterized by a number of data considerably lower than the number of the unknown parameters. This corresponds to a highly underdetermined system. To get a unique solution, a priori information must be therefore introduced. The simplest case is that of imposing that the norm of the solution be the smallest (minimum-length solution). In the case of potential field inversion, this leads to a distribution of the unknown density or susceptibility, which is very shallow and not representative of the true source distribution. In fact, requiring the solution to be small corresponds to finding a solution corresponding to the shallowest source distribution compatible with the measured data. This behavior was also recently explained by Silva et al. (2013) in terms of harmonic and biharmonic bias, forcing the solution to have maxima and minima on the borders of the modelized source region.

More realistic models of the distribution of the physical property at depth can however be obtained by introducing a “depth weighting” in the problem, able to counteract the natural decay of the kernel. Li and Oldenburg (1996) proposed to use a depth weighting function such as:

$$w(z) = \frac{1}{(z+q)^{\beta/2}} \tag{1}$$

where z is the depth of the layers and q depends on the height of survey. They suggested to use $\beta = 2$ in the gravity case and $\beta = 3$ in the magnetic case, corresponding to the fall-off rates of the field produced by a small cubic cell in the gravity and magnetic cases, respectively. Thus, Li and Oldenburg (1996) and Li and Oldenburg (1998) chose to ‘tune’ their depth-weighting function according to the power-law decay of the field produced by one single cell in the source domain. The Zhdanov (2002) and Čuma et al. (2012) inversion scheme uses integrated sensitivity weights changing according to the differentiation order. Cella and Fedi (2012) showed instead that the most appropriate value of β is equal to N, the structural index of the source. The structural index may be, in turn, estimated with standard methods such as Euler Deconvolution or the DEXP method (Fedi, 2007) and then introduced into the objective function.

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In this paper we will show the existence of invariance rules occurring in the inversion of potential fields of different orders, which may be useful in order to compare the source model distributions derived from the regularized inversion of gravity (or magnetic) fields with those from their derivatives.

2. The weighted minimum-length solution for the magnetic field and its *k*-order derivatives

Inverse potential-field problems are described by Fredholm integral equations of the first kind, which are, by nature, ill posed problems. For instance, the inverse geomagnetic problem has the following formulation (Blakely, 1996; Grant and West, 1965):

$$\Delta T(\mathbf{r}) = \int_{\Omega} A(\mathbf{r},\mathbf{r}_0)M(\mathbf{r}_0)d^3\mathbf{r}_0 \tag{2}$$

where: $\Delta T(\mathbf{r})$ is the measured anomalous magnetic field at the measurement point $\mathbf{r}(x,y,z)$; $M(\mathbf{r}_0)$ is the unknown distribution of magnetization at $\mathbf{r}_0(x_0,y_0,z_0)$; the integral is a volume integral over the volume Ω; and the kernel $A(\mathbf{r}, \mathbf{r}_0)$ is the field at $\mathbf{r}$ from a magnetic dipole of unit strength located at position $\mathbf{r}_0$. If we assume that the magnetization direction of the source is parallel to the direction of the main field and that both directions are vertical, $A(\mathbf{r}, \mathbf{r}_0)$ takes the following simple form:

$$A(\mathbf{r},\mathbf{r}_0) = \frac{3(z-z_0)^2}{\|\mathbf{r}-\mathbf{r}_0\|_2^5} - \frac{1}{\|\mathbf{r}-\mathbf{r}_0\|_2^3}. \tag{3}$$

Similar equations will occur for the gravity case and for any order derivative of either the gravity or the magnetic field. Here and throughout the paper, $\|\cdot\|_2$ denotes the vector two-norm.

Throughout this study the volume containing the source is assumed to be a rectangular volume of dimensions $d_x \times d_y \times d_z$. We also assume that the points at which the total-field anomaly is measured lie within a plane of dimensions $e_x \times e_y$ located at height h above volume Ω.

To solve the integral Eq. (2) numerically, we divide Ω into N rectangular cells arranged in an $N_x \times N_y \times N_z$ grid that covers Ω, and we assume a constant magnetization M_j within each cell Ω_j. This leads to a $P \times N$ system of linear equations of the form:

$$\mathbf{d} = \mathbf{Am} \tag{4}$$

where $\mathbf{d}$ is the column vector of the observation data (magnetic or gravity field), $\mathbf{m}$ is the column vector of the unknown (susceptibility/magnetization or density) and $\mathbf{A}$ is the kernel rectangular matrix.

In this paper we will consider not only the classical inversion of gravity and magnetic field data, but also that of their derivatives $\mathbf{f}$ of some order k. In this case, our system will be written as:

$$\mathbf{f} = \mathbf{A}_f\mathbf{m}_f. \tag{5}$$

If the number of unknowns is much greater than the number of data, the simplest solution of system (4) is the so-called minimum-length solution, which is the solution of problem (4) corresponding to minimizing the objective function

$$\varphi_m = \|\mathbf{m}\|_2^2 \tag{6}$$

subject to the constraint $\mathbf{d} - \mathbf{Am} = 0$.

This solution (e.g., Menke, 1984) has the form:

$$\mathbf{m}=\mathbf{A}^{\mathbf{T}}\left(\mathbf{AA}^{\mathbf{T}}\right)^{-\mathbf{1}}\mathbf{d}. \tag{7}$$

If some information for the model is available, regarding for instance the source distribution at depth, the model objective function ϕ_m may incorporate this information, by using a specific weighting matrix $\mathbf{W}_m$ such as (e.g., Li and Oldenburg, 2003):

$$\mathbf{W_m} = diag\left(\frac{1}{(z_i+q)^{\beta}}\right) \qquad i = 1,\ldots,L \tag{8}$$

where β is an exponent to be fixed and q is a quantity dependent on the altitude of the measurements. So, problem (6) becomes that of minimizing:

$$\varphi_m = \|\mathbf{W_m m}\|_2^2 \tag{9}$$

subject to the constraint $\mathbf{d} - \mathbf{Am} = 0$.

The solution is called weighted minimum-length solution (Menke, 1984):

$$\mathbf{m} = \mathbf{W_m^{-1}A^T}\left[\mathbf{AW_m^{-1}A^T}\right]^{-\mathbf{1}}\mathbf{d}. \tag{10}$$

A similar expression is obtained for the weighted minimum-length solution of Eq. (5):

$$\mathbf{m}_f = \mathbf{W_m^{-1}A}_f{}^{\mathbf{T}}\left[\mathbf{A}_f\mathbf{W_m^{-1}A}_f{}^{\mathbf{T}}\right]^{-\mathbf{1}}\mathbf{f}. \tag{11}$$

The depth-weighting exponent is normally assigned as the decay-exponent of the gravity or magnetic field, due to a single block (Li and Oldenburg, 1996, 1998). The same concept was applied to the gravity field derivatives (Li, 2001). Accordingly, we can generalize the approach for any-order derivatives of potential fields: calling β the depth weighting exponent of the magnetic field, and β_k that of its k-order derivatives, we will have:

$$\beta_k = \beta + k. \tag{12}$$

Eq. (12) applies also to the gravity case, where β is the depth-weighting exponent of the gravity gradient components and β_k that of their k-order derivatives. Following Cella and Fedi (2012) an optimal choice for β is instead the structural index N of the investigated source-distribution. However, since N also increases by k with the order of differentiation of the field (e.g., Blakely, 1996), Eq. (12) holds again.

The aim of this paper is to show:

a) that Eq. (12) may lead to not adequate source models; and
b) that consistent models are instead obtained if we assume that the depth weighting exponent β relative to the magnetic field and to any of its derivatives of order k is the same:

$$\beta_k = \beta. \tag{13}$$

In order to demonstrate b) let us first consider that $\mathbf{A}f = \mathbf{DA}$, where $\mathbf{D}$ is any invertible linear operator matrix, such as the operator of directional differentiation. In principle, the differentiation operator sends all the constant functions to zero, so as to be not invertible. However, its inverse operator, the integration operator, is uniquely defined if we set the constant functions as to some arbitrary value and it is not singular. In potential field analysis these constant functions, often called field zero-level, are normally affected also by filtering and normal field subtraction. Nevertheless, the zero-level may be assessed under additional conditions on the whole field behavior or on the related source properties, in practice in the model appraisal step. Due to this, we can therefore consider the differentiation operator as an invertible one.